Quantum Gravity Renormalization Group

by

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Author's Declaration

I hereby declare that this thesis consists of material that I am either the sole author or co-author. See Statement of Contributions for details. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Statement of Contributions

This thesis is largely a compilation of articles [139], [140], [141], [49], [142], [26], [48] and [112]. [70] is also referred to in some of the chapters.

In particular, chapter 2 is based on articles [140] and [141], which are single author papers and results from [70] that was co written with Henrique Gomes also features.

Chapter 3 is based on [139], [142] which were also single author and [26] which was co-written with Pawel Caputa and Shouvik Datta.

Chapter 4 is based off of [49], which was written in collaboration with William Donnelly.

Finally chapter 5 is based on [48] which was co-written with William Donnelly, Elise LePage, Andre Pereira, Yanyan Li and [112] which was co-written with Edward Mazenc and Ronak Soni.

Abstract

Investigations relating the emergence of the radial direction in the holography, solvable irrelevant deformations of conformal field theories and holographic entanglement entropy are reported. Emphasis is laid on how diffeomorphism invariance in the bulk emerges, in both the classical and quantum regimes. The computation of von Neumann entanglement and Renyi entropies in $T\bar{T}$ deformed conformal field theories and their bulk dual quantities are presented. Further, the peculiar properties of the deformed theory that these quantities point to, are discussed. Finally, the connections between $T\bar{T}$ deformed partition functions and solutions to the radial Wheeler de Witt equation in three spacetime dimensions is presented.

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I would like to start by thanking my supervisor Lee Smolin for his guidance, patience, and wisdom which were essential for carrying me through my PhD. I would then like to thank Sung-Sik Lee and Laurent Freidel for imparting their to me their unique styles of scientific inquiry. I would like to thank Bianca Dittrich for the many illuminating and educational lunch discussions. I would also like to thank Rob Myers for his advice and guidance at the crucial juncture of my PhD.

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Dedication

To my many mentors.

Table of Contents

List of Figures

Chapter 1

Introduction

1.1 Overview

The focus of this thesis is understanding the holographic principle as it applies to spacetime regions of finite volume. The principle states that the description of gravitational physics in some region of space is encoded entirely on the lower-dimensional boundary of the region [88],[146]. Among the original motivations for proposing this principle were to explain why a black hole's entropy is proportional to the area of its horizon. Similarly, one could hope to explain this way why the cosmological horizon surrounding any observer in a space with positive cosmological constant too has an entropy that scales with its area.

In either the case of black holes in the physical universe or in case of the cosmological horizon, what isn't understood is the precise mechanism behind how the degrees of freedom are localized to the boundary of the relevant spacetime regions, and what the dynamics of these degrees of freedom are. The so-called Anti-de Sitter(AdS)/Conformal Field theory(CFT) correspondence [111] is, however, a setting where such a mechanism is very well understood. This correspondence or duality equates gravitational physics in a space of negative cosmological constant and a quantum field theory that enjoys conformal symmetry living on a one lower-dimensional, asymptotic boundary of this space. The correspondence comes with a dictionary [74], [154] between observables on either side of the duality, and therefore quantities computed in one setting can be translated into dual quantities in the other. The hope is that an understanding of the robust features of the duality in the AdS/CFT setting will carry over into those of physical relevance mentioned above.

This thesis deals in particular with the matter of whether this duality is robust towards a change in the asymptotic structure of the bulk spacetime. In particular, the setting of interest is the situation where the boundary of the anti-de Sitter space is not infinitely distant from points in the interior. Instead, a finite boundary truncates the AdS space, and on it are imposed Dirichlet boundary conditions for any fields in the bulk. This is then a region of finite spacetime volume, just like the interior of a black hole behind its event horizon or the region behind a cosmological horizon in de Sitter space. Therefore studying holography in this context is a step in the direction of applying it to the physically relevant settings mentioned above.

The theory that lives on the boundary of this finite volume region involves deforming the conformal field theory that would have inhabited the asymptotic boundary by a special class of irrelevant operators. In particular, these will be the $T\bar{T}$ deformation [143],[32] of two dimensional conformal field theory and it's higher-dimensional cousins, which will be referred to as T^2 deformation [147],[80].

These deformed theories possess many interesting properties of pure quantum field theoretic interest as well. On the space of quantum field theories, the deforming operators trigger a flow away from the conformal fixed point but are still contained in the critical surface on which the fixed point lies. This is due to their irrelevant nature. Following them away the fixed point leads us to theories that have very different locality properties. In this thesis, some of these field-theoretic questions will also be investigated.

Returning to holography, we note that the AdS/CFT correspondence points towards an identification between the direction normal to the boundary (also called the radial direction) and the energy scale associated with the renormalization group flow of the quantum field theory living on the boundary. Then the renormalization group flow should map to development along the radial direction. This is a bulk counterpart of the flow triggered by the irrelevant deformations mentioned in the previous paragraph. Although holographic RG flows [40] have been studied extensively in the past, it wasn't quite understood how to introduce breaking of conformal invariance in the field theory that mirrors the effect of a sharp bulk radial cutoff surface.

On the theme of probing the locality properties of these quantum field theories, we will also focus on the study of entanglement entropy. An important lesson the AdS/CFT correspondence has taught us is the surprising connection between quantum information theory and spacetime physics, which started with the study of a quantity known as holographic entanglement entropy [137]. In the vein of learning broader lessons of holography, one can ask whether these connections persist in the less idealized setting of finite spacetime regions and the corresponding quantum theories on their boundaries. In so doing, the peculiar and distinctive features of the quantum field theories inhabiting these finite boundaries will be uncovered.

1.2 Themes and motivations

In this section, we will provide a cursory description of some of the broad questions that the thesis deals with. These descriptions will contain very few technical details, but they will appear in the coming chapters.

1.2.1 Gravity and Effective Field theory

General Relativity is our most successful attempt at describing gravitational physics on a variety of scales. It has received and continues to receive experimental validation in a variety of experiments and observations. An interesting question to ponder is what picks out general relativity among other theories of a dynamical metric. Lovelock's theorem [109] answers this question. The statement of the theorem is the only possible second-order Euler-Lagrange equation obtainable in a four-dimensional space from a scalar density of the form $\mathcal{L}(h)$ is

$$E^{AB} = \alpha\left(R^{AB} - \frac{1}{2}g^{AB}R\right) + \lambda h^{AB} = 0, \tag{1.1}$$

where α and λ are constants. In dimensions higher than four, there are so called quasi-topological theories whose actions are given by:

$$S = \int \mathrm{d}^d x\sqrt{g}\left(\delta^{A_1\cdots A_{2d}}_{B_1\cdots B_{2d}} R^{B_1 B_2}_{A_1 A_2}\cdots R^{B_{2d-1}B_{2D}}_{A_{2d-1}A_{2d}}\right) \tag{1.2}$$

where

$$\delta^{A_1\cdots A_{2d}}_{B_1\cdots B_{2d}} = (2d)!\delta^{A_1}_{[B_1}\cdots\delta^{A_{2d}}_{B_{2d}]}. \tag{1.3}$$

The equations of motion obtained from these actions are also second order in derivatives. So in a general number of dimensions, the requirement of general covariance, specifying that the metric is the only dynamical field, and restricting our attention to actions that generate equations of motion that are second order in derivatives doesn't suffice to pick out general relativity.

By seeking a parallel theorem in the Hamiltonian framework, this ambiguity can be circumvented. Whenever we ask about a Hamiltonian formalism, we need to introduce a foliation of spacetime by codimension one surfaces. For general relativity too, as first addressed in [11], this is indeed the procedure. For further details regarding the $D+1$ split and the Hamiltonian formalism, see appendix A.2.1. The phase space variables here are the metric induced on the codimension one slices

$$g_{\mu\nu} = h_{\mu\nu} \pm n_\mu n_\nu, \tag{1.4}$$

where n_μ is the one form dual to the normal vector to the hypersurfaces. Say the hypersurfaces are level sets of a function r, then the normal vector is related to the vector field

$$(\partial_r)^A = N n^A + \xi^A \tag{1.5}$$

where N is called the lapse function and ξ^A is the shift vector. The latter has components only tangential to the leaves of the foliation.

We also have its momentum conjugate $\pi^{\mu\nu}$ which is related to the extrinsic curvature of the surface in the following manner:

$$K^{\mu\nu} = \frac{2N}{\sqrt{g}} \left(\pi^{\mu\nu} - g^{\mu\nu}\mathrm{tr}\pi\right). \tag{1.6}$$

The Hamiltonian is a sum of constraints:

$$H_{Tot.} = \int \mathrm{d}^D x \left(N(x)H(g,\pi) + \xi^\mu H_\mu(g,\pi)\right). \tag{1.7}$$

Before discussing the explicit form of the constraints $H(g,\pi)$ and $H_\mu(g,\pi)$ in general relativity. Introducing the notation

$$H(N) = \int \mathrm{d}^D x N(x) H(g,\pi) \tag{1.8}$$

$$H_\mu(\xi^\mu) = \int \mathrm{d}^D x \xi^\mu(x) H_\mu(g,\pi). \tag{1.9}$$

Before looking at the form of the functions $H(g,\pi)$ and $H_\mu(g,\pi)$, let's ask- what could their forms possibly be given that we are describing the dynamics of hypersurfaces covariantly embedded in an ambient spacetime. The criterion for the embeddibility is encoded in the Poisson algebra of these constraints, and the reason for this will be explained in the following chapter. This Poisson algebra is given by:

$$\{H(\sigma), H(\sigma')\} = -H_\mu(f^\mu(\sigma,\sigma')),\ \{H_\mu(\xi^\mu), H(\sigma)\} = H(\xi^\mu\partial_\mu\sigma),$$

$$\{H_\mu(\xi^\mu), H_\nu(\zeta^\nu)\} = H_\mu([\xi, \zeta]^\mu). \tag{1.10}$$

The interesting result due to Hojman, Kuchar and Teitelboim [138] is that this form of the Poisson algebra, given the phase space variables directly implies that the constraints must be those of general relativity, found by Arnowitt, Deser and Misner:

$$H(N) = \int \mathrm{d}^D x N(x) \left(\frac{1}{\sqrt{g}} \left(\pi_{\mu\nu}\pi^{\mu\nu} - \frac{1}{D-1}\mathrm{tr}\pi^2 \right) - \sqrt{g}(\Lambda - R) \right) \tag{1.11}$$

$$H_\mu(\xi^\mu) = \int \mathrm{d}^D x \xi^\mu \nabla_\rho \pi^\rho_\mu. \tag{1.12}$$

More will be made of this theorem and it's implications in the coming chapters, but for now, we see that the Hamiltonian of general relativity, which contains as much information as the Einstein equations, were picked out uniquely by specifying the phase space and the gauge invariance in the form of the Poisson algebra of constraints.

That the phase space variables are just the metric and the momentum conjugate mirrors the statement that there are at most two derivatives of the metric in the action. However, this statement can only be made precise for the derivatives of the metric w.r.t r.

The reason the quasi topological theories aren't picked out is that we implicitly restrict our attention to single-valued Hamiltonian flows. In the quasi topological case, the fact that the relationship between the r derivatives of the metric induced on the hypersurface and the momentum is not invertible, so in trying to find a Hamiltonian, it will inevitably lead to a flow on phase space that branches.

In summary, if we pose the question: what theory of evolving D dimensional geometries describe a theory of spacetime? then general relativity is the unique answer.

What it means for a direction of space to emerge: Let's focus on the setting where the theory in the bulk is classical. The holographic dictionary identifies bulk fields with boundary sources, and this in turn implies that the equations of motion of the bulk are in correspondence with the renormalization group equations of the boundary theory. The variant of the renormalization group which can capture these dynamics is the so-called local renormalization group [126]. This will be described in the coming chapter. Here, sources are introduced for the composite operators of interest in the field theory and their scale evolution is captured by equations of the form

$$\frac{d}{dz} J^I(x, z) = -\beta^I(J). \tag{1.13}$$

Here $\beta^I(J)$ are the local analogs of the beta functions. Similarly, one can track the scale evolution of the expectation value of the operators to which these sources couple:

$$\frac{d}{dz}\langle O_I\rangle = \Gamma_I^J\langle O_J\rangle + \cdots \tag{1.14}$$

Here, Γ_J^I are analogous to the anomalous dimensions. These equations will further be explained in the coming chapter, but we already see what the puzzles might be if one is interested in theories whose local RG equations can mimic bulk equations of motion.

First, the question to ask is whether these equations can replicate second-order equations of motion, that we are most accustomed to dealing with in theories of gravity coupled to various fields. The answer to this is will lie in noting that the above equations resemble Hamilton's equations more than they do Euler Lagrange equations, and should be cast as such. Furthermore, how a Hamiltonian that has the right number of momenta to replicate that of a theory whose Euler Lagrange equations are second-order will be addressed by the use of the quantum renormalization group. For the present discussion. Say we are granted such a Hamiltonian. Then, what remains is to ensure that the Hamiltonian so obtained is formed from constraints, and that the Poisson algebra mirrors that of general relativity. Note that this is a radial Hamiltonian describing evolution along a space-like direction. We will also mostly be interested in Euclidean field theories in the bulk.

Now, restricting our attention to the metric as a source, we see that we are in the situation where a phase space is granted to us, and a Hamiltonian whose form we do not know. However, if we demand that the Poisson algebra of constraints mimics that of general relativity, then by the HKT theorem, we are guaranteed general relativity emerges in the bulk. In other words, here too we see that we are to choose from theories of evolving geometries and ask which ones describe a covariant theory in spacetime. The renormalization group equivalent to the statement of the Poisson algebra will be a central focus of this thesis and will be shown explicitly in the coming chapter. This renormalization group parallel to the Poisson algebra will be a particular way in which the Wess–Zumino consistency conditions need be satisfied. It will be called the Holographic Wess–Zumino consistency condition. Its implications will echo throughout the thesis.

To put very plainly, we are interested in how a holographic theory must respond to scale transformations so that the scale direction that emerges is treated on equal footing as the other spatial directions. Thus, to answer the question posed in the title of this subsection, for a direction of space to emerge, it must happen in a way that respects the diffeomorphism invariance in the bulk. A way to see what is happening here is that the freedom to choose local frames of reference in the bulk translates to the freedom to specify local rates of coarse-graining and choose different coordinate systems when considering the coarse-grained theory.

1.2.2 Quantum Theory in the Bulk

One of the opportunities that holography affords us is a look at what could take over from general covariance at the quantum level. If general relativity is to emerge, then all we can say about diffeomorphism invariance is that it ought to be instated in the regime where classical general relativity is valid, but beyond that, whatever is to become of it depends crucially on what takes over from the theory of general relativity at the quantum level. In particular, there isn't any reason to believe that the invariance itself will remain intact in the quantum theory. In the setup at hand, we shall have a renormalization group parallel to the statement of general covariance at hand. This dual condition is identified in the regime where the bulk theory is classical. Then, we need only consider the regime of the quantum field theory in which the bulk dual is not classical. Since the consistency condition of the renormalization group can still be phrased in this regime, it will tell us how diffeomorphism invariance manifests at the quantum level.

In the final chapter of this thesis, we will consider quantum gravity in three dimensions with a negative cosmological constant. The dual theory is the $T\bar{T}$ deformation of a two-dimensional conformal field theory, and as such, the flow equation defining that theory holds at a finite central charge and mimics the Wheeler de Witt equation. In this situation, the consistency conditions that are dual to the commutator algebra of the constraints are of the same form as in the classical regime. (This is, provided we replace Poisson brackets with commutators)

1.2.3 Holographic entanglement entropy with finite boundaries

In holographic entanglement entropy calculations in conventional AdS/CFT, the leading divergences in the entanglement entropy in the CFT are matched by the divergent volumes of the codimension two surfaces anchored to the entangling surface at the asymptotic boundary. If we consider the finite volume regions of interest here, were we to imagine that the volume of co-dimension two surfaces anchored to the boundary, it will always remain finite. The question then is, whether such a quantity is dual to entanglement entropy in the deformed theory living on the boundary at finite radius. This question is answered in the affirmative in the third chapter of this thesis. This finiteness of the entanglement entropy is an indicator that perhaps the theory at hand is somehow equivalent to a quantum system with a finite-dimensional Hilbert space. Further, in the final chapter, we investigate whether these calculations can be carried out when the bulk theory is no longer classical as well.

1.3 Plan of the Thesis

In chapter 2, the aforementioned issue of how bulk diffeomorphism invariance emerges will be addressed. The discussion starts with introducing the quantum renormalization group, and within this context, the requirement of bulk covariance is seen to lead to the Holographic Wess–Zumino consistency conditions. Connections are made to some conventional results in holographic renormalization. In chapter 3, flows that satisfy the holographic WZ conditions are discussed. The chapter begins with applying QRG to two-dimensional holographic CFTs deformed by the $T\bar{T}$ to construct the bulk Einstein gravity theory in AdS_3. Then, higher dimensional T^2 deformations, and the connections between the flows they trigger and the generalized gradient flow is discussed, and here too, some connections to conventional holographic renormalization are made. Then chapter 4 addresses the computation of von Neumann and Renyi entanglement entropy in $T\bar{T}$ deformed CFTs at large central charge. It is shown that the holographic entanglement entropy conjectures generalize to the case where the bulk is cutoff at a finite radius. Finally, in chapter 5, the connections between the partition function of $T\bar{T}$ deformed CFTs at any finite charge and the solutions to the radial Wheeler de Witt equation is described. The computation of von Neumann entropy at a finite central charge in the $T\bar{T}$ deformed theory is also presented.

Chapter 2

Emergent Bulk diffeomorphism invariance

As described in the previous chapter, a key feature of the holographic duality is the identification between the radial direction in the bulk and the energy scale of the quantum field theory on the boundary. If the bulk theory is generally covariant, then it must be that the energy scale and the directions of space on which the field theory lives must be treated on equal footing. In other words, full bulk diffeomorphism invariance must be encoded in the renormalization group flow of the holographic field theory. This is encoded in a particular form of the Wess–Zumino consistency condition that the local renormalization group must satisfy.

In this chapter, these conditions will be presented in the context of the quantum renormalization group(QRG). This is a constructive prescription for deriving bulk theories from re-organizing the RG flow of certain quantum field theories.

2.1 The Local RG Flow

The most general notion of coarse-graining available in real space which remains meaningful even on arbitrary backgrounds is the one given by local Weyl transformations of the background metric. The renormalization group flow can be seen as the response of the generating functional under such a change. This perspective is known as the local renormalization group. It is a continuum generalization of Kadanoff's idea of block spin transformations. This approach was pioneered by Osborn in [126].

The usual re-scaling transformations associated to the scale evolution can be seen as Weyl transformations with a constant Weyl factor:

$$\frac{\partial \mathrm{ln} Z}{\partial \tau} = \int \mathrm{d}^D x \sqrt{g} g^{\mu\nu} \frac{\delta \mathrm{ln} Z}{\delta g^{\mu\nu}(x)},$$

and that more generally, one can have transformations with an arbitrary Weyl parameter:

$$\delta_\sigma = \int_x \sigma(x) g^{\mu\nu} \frac{\delta}{\delta g^{\mu\nu}},$$

and then, ask how the generating function responds:

$$\delta_\sigma \mathrm{ln} Z = \int \mathrm{d}^D x \sqrt{g} \sigma(x) \beta^{\tilde{\alpha}}(J) \frac{\delta \mathrm{ln} Z}{\delta J^{\tilde{\alpha}}(x)}. \tag{2.1}$$

Here, the index set $\{\tilde{\alpha}\}$ includes the identity operator and the metric. Thus, the term $U(J)$ also appears on the right hand side.

One can then change scheme to include a term proportional to $g^{\mu\nu}$ in the definition of the metric beta function, i.e. $\beta^{\mu\nu} \to \beta^{\mu\nu} + g^{\mu\nu}$, and in the new scheme, the above equation reads

$$\Delta_\sigma \mathrm{ln} Z \equiv \int \mathrm{d}^D x \sqrt{g} \sigma(x) \beta'^{\tilde{\alpha}} \frac{\delta \mathrm{ln} Z}{\delta J^{\tilde{\alpha}}} = 0. \tag{2.2}$$

The beta functions $\beta'^{\tilde{\alpha}}$ are those that include the metric as per the aforementioned scheme change. This is known as the local Callan–Symanzik equation.

2.2 Mapping RG flow equations to Hamilton's equations

A very useful insight to this thesis is that of Dolan in [43], which is that RG flow equations can be mapped to Hamilton's equations. The phase space variables are identified as follows. The source $J(x)$ is taken to be the configuration variable, and its canonical conjugate momentum is the one-point function of the corresponding operator:

$$\langle \mathcal{O}_\alpha(x) \rangle_J = \frac{\delta \mathrm{ln} Z}{\delta J^\alpha(x)}. \tag{2.3}$$

Note that this one point function is computed in presence of a general source.

The symplectic form on this phase space reads

$$\Omega = \int \mathrm{d}^D x \sqrt{g} \delta \langle \mathcal{O}_\alpha(x) \rangle \wedge \delta J^\alpha(x). \tag{2.4}$$

The generating functional $W[J^\alpha(x), g_{\mu\nu}(x)] = \ln Z[J^\alpha(x), g_{\mu\nu}(x)]$ plays the role of Hamilton's principal function, or the on shell action. Hence, the symplectic form evaluated on shell vanishes:

$$\Omega|_{\langle \mathcal{O} \rangle = \frac{\delta W}{\delta J}} = 0.$$

For notational convenience, henceforth we introduce the notation, $P_\alpha(x) \equiv \langle \mathcal{O}_\alpha(x) \rangle$. In the case of the renormalised theory, these identifications were first made by Dolan in [43]. This identification for the regularised theory was also made in [130].

The simplest demand one can make of the Hamiltonian which generates scale evolution and hence drives the renormalization group flow is that Hamilton's equations are the flow equations:

$$\frac{\partial J^\alpha}{\partial \tau} = \beta^\alpha(J), \tag{2.5}$$

$$\frac{\partial P_\alpha}{\partial \tau} + \frac{\partial \beta^\sigma}{\partial J^\alpha} P_\sigma = -\frac{\partial U(J)}{\partial J^\alpha}. \tag{2.6}$$

The first equation is the renormalization group flow equation describing the scale evolution of the sources, whereas the second equation is the equation describing the flow of the one-point functions in the presence of sources. The function $U(J)$ in the latter equation encodes the flow of the coupling of the identity operator, which is also necessary to include.

Upon inspection, one sees that the Hamiltonian which satisfies these demands takes the form

$$H[J, P] = \int \mathrm{d}^D x \sqrt{g} (\beta^\alpha(J) P_\alpha - U(J)). \tag{2.7}$$

The relationship between this Hamiltonian and the generating functional reads

$$H = \frac{\partial}{\partial \tau} \ln Z = 0. \tag{2.8}$$

This is the usual statement regarding the RG invariance of the generating functional. This fact implies that the Hamiltonian is zero and hence we are dealing with a completely constrained system. In particular, this is reminiscent of a re-parameterization invariant system. Such systems are invariant under the relabelling $\tau \to f(\tau)$. It is of value to introduce a lapse function $\sigma(\tau)$ which compensates for such changes so that $\sigma \mathrm{d}\tau$ remains invariant. This means that under the re-parameterization transformation $\sigma \to f'(\tau)^{-1}\sigma$. It appears in the action as the Lagrange multiplier enforcing the vanishing of the Hamiltonian constraint. It will be convenient to define

$$H(\sigma) = \int \mathrm{d}^D x \sqrt{g}\sigma(\beta^\alpha(J)P_\alpha - U(J)). \tag{2.9}$$

Since we are interested in the local Renormalization group, we generalize this setup so that our Hamiltonian constraint is:

$$H(\sigma) = \int \mathrm{d}^D x \sqrt{g}\sigma(x,\tau)\beta'^{\tilde{\alpha}}(J(x,\tau))P_{\tilde{\alpha}}(x,\tau). \tag{2.10}$$

This is a re-writing of equation (2.2), which, as mentioned before is a statement of local RG invariance of the generating functional. It will be convenient though to separate the term corresponding to the flow of the coupling of the identity operator:

$$H(\sigma) = \int \mathrm{d}^D x \sqrt{g}\sigma(x,\tau)\left(\beta^{\tilde{\alpha}}(J(x,\tau))P_{\tilde{\alpha}}(x,\tau) - U(J)\right).$$

Notice that this Hamiltonian is linear in the momenta, and so it follows that Hamilton's equations are first order. To mimic nontrivial equations of motion of some bulk theory, we would like second or higher-order equations of motion. This will require that the Hamiltonian have a kinetic term that's at least quadratic in momenta. The quantum renormalization group applied to large N matrix field theories provides us with such Hamiltonians, as we shall see.

2.3 Quantum RG

The Quantum Renormalization group (QRG) due to Sung-Sik Lee [103], [102], is a constructive coarse-graining prescription used for finding holographic dual descriptions of quantum field theories. This mechanism will be explained in this thesis through the example of large N matrix field theories.

2.3.1 Quantum RG applied to large N Matrix field theory

The fundamental fields are Hermitian $N \times N$ matrix fields $\Phi(x)$ (we will suppress the matrix indices for notational ease). Assume also that the matrix model is gauge under some gauge group ($U(N)$ for instance), this means that gauge invariant operators in the action are necessarily sums and products of traces of monomials of the matrix fields and their derivatives. Note that this was first worked out in [102] and similar results were derived in [3] and in [18] under the name of the 'planar Polchinski equation'.

The simplest gauge invariance operators are the so called single trace operators that are of the form:

$$O_{\{m\}} = \frac{1}{N}\sqrt{g}\mathrm{tr}\left(\Phi(\nabla_{\mu_1^1}\cdots\nabla_{\mu_{p_1}^1}\Phi)\cdots(\nabla_{\mu_1^q}\cdots\nabla_{\mu_{p_q}^q}\Phi)\right), \tag{2.11}$$

where the multi index set $\{\mu_{p_i}^i\}$ is used to denote the fact that there can be varying number of derivatives and arbitrary permutations of indices thereof in each term of the product of derivatives in the above operator. For brevity, we will use just a single latin letter index m to encapsulate the multi-index set above. Multi-trace operators are formed from products and derivatives of products of single trace operators. The sources for single trace operators will be denoted as J^m and for multi trace operators as $\mathcal{J}^m$. These sources have arbitrarily many indices and in particular, the source for the term in the action with two derivatives can be chosen to be the background metric.

The action reads:

$$S = S_o[\Phi(x)] + N^2\sum_m \int \mathrm{d}^3x\sqrt{g}\,\underbrace{J^m(x)O_m}_{Single-Trace} + \sum_m \int \mathrm{d}^3x\sqrt{g}\,\underbrace{V_m[O_m, \mathcal{J}^m]}_{Multi-Trace}. \tag{2.12}$$

For reasons which will be made clear in the remainder of this sub-section, it will be of interest to study the renormalization group trajectories starting from the sub space of single trace operators. The generating functional for a theory on this sub space is given by

$$Z[J^m] = \int \mathcal{D}\Phi \exp i\{S[O_m(\Phi); J^m]\}. \tag{2.13}$$

Wilsonian RG describes how the sources(/couplings) change when the UV cutoff Λ is lowered by a factor[1] $\sigma(x) \equiv \alpha(x)\delta z$:

$$\Lambda \to e^{\alpha(x)\delta z}\Lambda,$$

through the equations defining beta functions β^m for sources $J^m(x)$. Solutions to those first-order ODE's will determine a path in the space of sources/couplings. As mentioned before, when the theory is strongly coupled, there are, in general, infinitely many such sources. In that case, solving all RG flow equations becomes intractable.

Now, in the limit as $N \to \infty$, the quantum corrections to the single trace action under one step of RG ($\Lambda \to \Lambda - \Lambda\delta z\alpha(x)$) take the form[2] :

$$\delta S[O_n, J^n] = N^2 \int \mathrm{d}^D x\sqrt{g}\alpha(x)\delta z(\mathcal{L}_C(J^n(x)) - \beta^m(J^n(x))O_m +$$

$$\frac{G^{mn\{\mu\}\{\nu\}}(J^n(x))}{2}\nabla_{\{\mu\}}O_m\nabla_{\{\nu\}}O_n) + \mathcal{O}(\delta z^2) \tag{2.14}$$

Here, the term $\mathcal{L}_C(J^n(x))$ is the integrand within the anomaly term denoted in the previous subsection as $\mathcal{A}_\sigma$, *i.e.* $\int \mathrm{d}^D x\sqrt{g}\delta z\alpha(x)(\mathcal{L}_C(J^n(x)) \equiv \mathcal{A}_{\sigma=\alpha(x)\delta z}$.

The value of applying quantum RG to the matrix field theory will become apparent here because it cleverly re-organizes the renormalization of this theory without having to ever leave the subspace of single trace operators. The honest fixed point of the flow cannot be projected down to this subspace because multi-trace operators are generated by quantum corrections. QRG is a method to nevertheless restrict the RG trajectory to this subspace and thereby project down the fixed point, by paying the price of promoting the sources in this subspace to fluctuating quantum fields. This is why it is apt to call this technique the quantum renormalization group. What is meant by the above statement will be fleshed out schematically in what follows.

The most important thing to notice about the quantum corrections to the single trace action is that to linear order in δz, only double trace operators are generated.

[1]This infinitesimal 'RG time step' δz is introduced to keep track of how many iterations of infinitesimal local RG transformations one performs.

[2]the indices in braces such as $\{\mu\}, \{\nu\}$ etc. of differential operators and tensors denote multi index sets distinct from those incorporated into the indices m, n and will be used solely in situations where a differential operator of arbitrarily high order is involved. For instance, $A^{m\{\mu\}}\nabla_{\{\mu\}} \equiv \sum_{k=0}^{k_m} A^{m\mu_1\cdots\mu_k}\nabla_{\mu_1}\cdots\nabla_{\mu_k}$, and k_m could in principle be infinity.

Toy integral example

Making precise the statement about the single trace sources being promoted to dynamical fields lies in the observation that the multi-trace operators (double trace in the planar limit) can be removed by paying the price of functional integration over auxiliary fields. To schematically describe this, we will take the example of an ordinary (as opposed to functional) integral over a single variable for simplicity. The main idea will carry through into the case of interest. Let the toy integrand to be transformed be of the following exponential form:

$$\int \mathrm{d}\varphi e^{iJO} = Z(J) = \int \mathrm{d}\varphi \, e^{i\left\{JO(\varphi)+\alpha\delta z(\beta(J)O(\varphi)+G(J)O^2(\varphi)+l_C)\right\}} \tag{2.15}$$

Notice that each of these terms is analogous to the functional integrand of the single trace planar matrix field theory. The variable φ now plays the role of the fundamental field and the source for single trace operators $O(\varphi)$ is the variable J. The first term in the exponential is analogous to the single trace action, the second term is analogous to the single trace beta function term and the third to the double trace term. The latter two terms are generated by quantum corrections in one step of RG, in addition to the renormalization of the coupling of the identity or the 'cosmological constant' term l_C, and hence a factor of $\alpha\delta z$ is retained as a reminder of this fact.

The goal is to remove the quadratic term in the exponent in the integrand at the cost of introducing integration over a new 'field' $p^{(1)}$:

$$e^{i\left\{JO(\varphi)+\alpha\delta z(\beta(J)O(\varphi)+G(J)O^2(\varphi)+l_C)\right\}} = \int \mathrm{d}p^{(1)}\delta(p^{(1)}-O)e^{i\left\{-Jp^{(1)}-\alpha\delta z(\beta(J)p^{(1)}-p^{(1)}G(J)p^{(1)}+l_C)\right\}}. \tag{2.16}$$

The delta function itself can be represented in integral form over another variable $j^{(1)}$:

$$\begin{aligned}
&\int \mathrm{d}p^{(1)}\delta(p^{(1)}-O)e^{i\left\{-Jp^{(1)}-\alpha\delta z(\beta(J)p^{(1)}-p^{(1)}G(J)p^{(1)}+l_C)\right\}} \\
&= \int \mathrm{d}p^{(1)}\mathrm{d}j^{(1)}e^{i(p^{(1)}-O)j^{(1)}}e^{i\left\{-Jp^{(1)}-\alpha\delta z(\beta(J)p^{(1)}-p^{(1)}G(J)p^{(1)}+l_C)\right\}} \\
&= \int \mathrm{d}p^{(1)}\mathrm{d}j^{(1)}e^{ip^{(1)}(j^{(1)}-J)}e^{i(j^{(1)}O)}e^{i\left\{-\alpha\delta z(\beta(J)p^{(1)}-p^{(1)}G(J)p^{(1)}+l_C)\right\}},
\end{aligned} \tag{2.17}$$

and if the $p^{(1)}$ integral were now performed, one would see that the above expression is hiding a delta function $\delta(j^{(1)}-J)$, which becomes apparent if the above expression is written as:

$$\int \mathrm{d}p^{(1)}\mathrm{d}j^{(1)}e^{ip^{(1)}(j^{(1)}-J)}e^{i(j^{(1)}O)}e^{i\left\{-\alpha\delta z(\beta(J)p^{(1)}-p^{(1)}G(J)p^{(1)}+l_C)\right\}}.$$

The fields $j^{(1)}$ and $p^{(1)}$ are denoted in a manner which suggests tentatively that they shall be related to the fluctuating sources and their conjugate vacuum expectation values on the RG phase space. The nature of this relation will be made explicit in what follows.

So now the 'partition function' can be written as

$$Z(J) = \int \mathrm{d}p^{(1)}\mathrm{d}j^{(1)} e^{ip^{(1)}(j^{(1)}-J)} e^{i\left\{-\alpha\delta z(\beta(J)p^{(1)}-p^{(1)}G(J)p^{(1)}+l_C)\right\}} Z(j^{(1)}), \tag{2.18}$$

where $Z(j^{(1)}) = \int \mathrm{d}\varphi \exp i\{jO\}$, and is thus the same as $Z(J)$ except that J is replaced by $j^{(1)}$. The RG transformations still involve coarse graining with respect to the fundamental fields φ and so since the quantum corrections have all been factored out into the integral in front of $Z(j^{(1)})$, the second step of RG will proceed exactly the way the first step did except that J is now replaced by the source $j^{(1)}$. Given this guarantee of maintenance of the form of the quantum corrections, one once again may choose to integrate in auxiliary fields $(p^{(2)}, j^{(2)})$ to obtain a result similar to the one above, *i.e.*

$$Z(j^{(1)}) = \int \mathrm{d}p^{(2)}\mathrm{d}j^{(2)} e^{ip^{(2)}(j^{(2)}-j^{(1)})} e^{i\left\{-\alpha^{(2)}\delta z(\beta(j^{(1)})p^{(2)}-p^{(2)}G(j^{(1)})p^{(2)}+l_C)\right\}} Z(j^{(2)}). \tag{2.19}$$

Here, $\alpha^{(2)}$ is the analogue of the Weyl factor chose at the second step of RG, which is free to be chosen to be different from α. Thus a pattern emerges, and if so the result of iterating this procedure for k RG steps can be written as:

$$Z(J) = \int \prod_{i=0}^{k} [\mathrm{d}\alpha^{(i)}\mathrm{d}p^{(i)}\mathrm{d}j^{(i)}] \times$$
$$\left(e^{i\sum_{i=0}^{k} k\left\{\delta z\left(\frac{p^{(i)}(j^{(i)}-j^{(i-1)})}{\delta z}\right)-\alpha^{(i)}\left(\beta(j^{(i-1)})p^{(i)}+p^{(i)}G(j^{(i-1)})p^{(i)}+l_C\right)\right\}}\Big|_{(j^{(0)},p^{(0)})=(J,O)} \right) Z(j^{(k)}). \tag{2.20}$$

Note that the different Weyl factors at each step of RG denoted $\alpha^{(i)}$ are also integrated over in order to 'average' over all possible RG paths. The consequences of the path independence of this RG procedure will play a very important role in the remainder of this chapter.

The continuum limit of the above product of integrals can the be defined by sending $\delta z \to 0$, and defining $z = \epsilon \exp(k\delta z)$ as the so called 'radial' time. Here, ϵ denotes a short distance cutoff. The integration variables then become: $\left\{\alpha^{(i)}, j^{(i)}, p^{(i)}\right\} \to \{\alpha(z), j(z), p(z))\}$. The latter two parameterise the dynamical phase space alluded to in earlier discussions. Then the set of integrals over all RG steps can be recast as the functional integral:

$$Z(J) = \int \mathcal{D}\alpha(z)\mathcal{D}j(z)\mathcal{D}p(z)e^{i\int_{z=0}^{z=z_*}\mathrm{d}z\left(p(z)\frac{\mathrm{d}j(z)}{\mathrm{d}z}-\alpha(z)H_{QRG}\right)}|_{(j(0),p(0))=(J,O)}Z(j(z_*)) \quad (2.21)$$

These 'fields' now gaining additional dependence on the RG time is the signature of the emergence of a new directions of space, or in other words, of holography. The field theory of these sources is holographically dual to the original theory of the φ fields. The bound on the integral z_* denotes where the RG transformations are truncated, which needn't necessarily be infinity. The function H_{QRG} is then give by

$$H_{QRG} = \beta(j(z))p(z) + p(z)G(j(z))p(z) + l_C. \quad (2.22)$$

As promised, the Hamiltonian is now quadratic in the 'momenta' $p(z)$ that are conjugate to the dynamical source variables $j(z)$.

Back to the Matrix Field Theory

Similarly, in the planar matrix field theory case, the Quantum Renormalization promotes the sources $J^m(x)$ and the vacuum expectation values of the single trace operators $\langle O_m(\Phi(x))\rangle$ to the dynamical fields $(j^m(x,z), p_m(x,z))$. In the large N limit, these single trace operators are equal to their vacuum expectation values. These fact that they are labeled by the RG time z in addition to the labels x is the precise sense in which they live in one dimension higher to the planar matrix fields. The Hamiltonian in that case reads:

$$Z[J^m] = \int \mathcal{D}\alpha(x,z)\mathcal{D}j^m(x,z)\mathcal{D}p_m(x,z)\times$$

$$\times e^{iN^2\int \mathrm{d}^Dx\mathrm{d}z\sqrt{g}\left(p_m(x,z)\frac{\mathrm{d}j^m(x,z)}{\mathrm{d}z}-\alpha(x,z)H_{QRG}(j^m(x,z),p_m(x,z))\right)}|_{(j^m(x,z=0),p_m(x,z=0))=(J^m(x),\langle O_m(x)\rangle)}Z[j^m]. \quad (2.23)$$

The factor of N^2 out in front of the integral in the exponent plays the role of $\hbar^{-1}$ for the partition function on the space of sources.

The large N limit is the same as taking the semiclassical limit of the bulk theory which in other words, allows the functional integral to be performed in the saddle point approximation. This saddle point corresponds to extremising the action $S_B = \int \mathrm{d}^Dx\mathrm{d}z\,(p_m\dot{j}^m - \alpha(x)H_{QRG})$. The Hamiltonian density takes the form:

$$H_{QRG}(j^m(x,z), p_m(x,z)) =$$

$$\beta^m(j^m(x,z))p_m(x,z) + G^{mn\{\mu\}\{\nu\}}\big(j^m(x,z)\big)\nabla_{\{\mu\}}p_m(x,z)\nabla_{\{\nu\}}p_n(x,z) + \mathcal{L}_C(j^m(x,z)). \tag{2.24}$$

The truncation to the single trace subspace is what leads to the quadratic term in the momenta and thus the truncation itself is responsible for the non trivial dynamics of these fields. The potential for the bulk fields is given by the generalisation of the term which renormalizes the coupling of the identity operator: $\mathcal{L}_C(j^m(x,z))$.

To conclude, the classical phase space in which QRG flow takes place is thus parameterised by the conjugate pairs $(j^m(x,z), p_m(x,z))$. This means that they satisfy the fundamental Poisson bracket relation

$$\{j^m(x,z), p_n(y,z)\} = \delta^m_n \delta(x,y). \tag{2.25}$$

The Hamiltonian generating this QRG flow is $H_{QRG}(j^m(x,z), p_m(x,z))$ given by (2.24). The phase space is thus a subspace (that of single race operators) of the one identified in the previous section, but the Hamiltonian now contains a term quadratic in the momentum.

2.3.2 Emergent gravity from QRG

The QRG procedure in the context of the matrix field theory will also promote the source of the single trace energy-momentum tensor, *i.e.* the metric to a dynamical field. As mentioned before, to study the pure gravity limit in the bulk, a limit where the energy-momentum tensor is the only operator in the theory with a finite scaling dimension needs to be considered. This can happen if all operators acquire large anomalous dimensions but the energy-momentum tensor is protected by its Ward Identity. This implicitly also requires the strong coupling on the planar matrix field theory's side, although the classicality of the bulk still requires the large N limit. It must also be assumed that there are no other conserved higher spin currents in the theory.

The phase space variables in this case will be the metric and the vacuum expectation value of the energy momentum tensor $(\pi_{\mu\nu}(x,z), g^{\mu\nu}(x,z))$, satisfying fundamental Poisson bracket relation

$$\left\{\pi_{\mu\nu}(x,z), g^{\alpha\beta}(y,z)\right\} = \delta_\mu^{(\alpha}\delta_\nu^{\beta)}\delta(x,y). \tag{2.26}$$

The bulk RG Hamiltonian then takes the form

$$\int \mathrm{d}^3x\sqrt{g}\left(\alpha(x,z)H(\pi_{\mu\nu}, g^{\mu\nu}) + \xi^\mu(x,z)H_\mu(\pi_{\mu\nu}, g^{\mu\nu})\right) =$$

$$\int \mathrm{d}^3x\sqrt{g}\alpha(x,z)\left(V(g) + \frac{G^{\mu\nu\alpha\beta\{\eta\}\{\rho\}}(g)}{2}\nabla_{\{\eta\}}\pi_{\mu\nu}\nabla_{\{\rho\}}\pi_{\alpha\beta} + \beta^{\mu\nu}(g)\pi_{\mu\nu}\right) +$$

$$+\int \mathrm{d}^3x\sqrt{g}\xi^\mu(x,z)\left(\nabla^\nu\pi_{\mu\nu}\right). \tag{2.27}$$

The first term is the RG Hamiltonian described in the previous section, and it generates local RG transformations. The Hamiltonian itself, (as opposed to the density) will be denoted as

$$H(\alpha) = \int \mathrm{d}^3x\sqrt{g}\alpha(x,z)H(\pi_{\mu\nu}, g^{\mu\nu}), \tag{2.28}$$

and for reasons mentioned in the previous paragraph, this is a constraint with the function α is the corresponding Lagrange multiplier. Diffeomorphism invariance of the matrix field theory arising due to being coupled to an arbitrary background also needs to be taken into account. This is captured by the Ward Identity:

$$\langle\nabla^\mu T_{\mu\nu}\rangle = 0, \tag{2.29}$$

that is imposed in the QRG as a constraint:

$$H_\mu(\xi^\mu) = \int \mathrm{d}^3x\sqrt{g}\xi^\mu\left(\nabla^\nu\pi_{\mu\nu}\right) = 0, \tag{2.30}$$

where the the shift vector ξ^μ is the Lagrange multiplier enforcing this constraint.

Thus the phase space of the bulk theory is that of general relativity in the Hamiltonian formalism (discovered by Arnowitt–Deser and Misner in [11]). Of course, the algebraic form of the scalar constraint is not quite that of general relativity. The question that will be addressed in the remainder of the chapter is under what circumstances the QRG scalar constraint becomes that of general relativity.

The functions $V(g)$, $\beta^{\mu\nu}(g)$ and $G^{\mu\nu\alpha\beta\{\eta\}\{\rho\}}(g)$ are not just functions of the metric but also its derivatives (*i.e.* curvature tensors) to arbitrarily high orders, but they admit a derivative expansion where the leading order terms are:

$$V(g) = -c_0 + c_1R + \cdots\,, \quad \beta^{\mu\nu}(g) = \beta g^{\mu\nu} + \cdots\,, \quad G^{\mu\nu\alpha\beta\{\eta\}\{\rho\}}(g) = \frac{\gamma}{g}G_\lambda^{\mu\nu\alpha\beta} + \cdots \tag{2.31}$$

Here, $G_\lambda^{\mu\nu\alpha\beta} = g^{\mu\alpha}g^{\nu\beta} + g^{\mu\beta}g^{\nu\alpha} - \lambda g^{\mu\nu}g^{\alpha\beta}$.

If one entertains the possibility of tuning the constants in the gradient expansion then it is conceivable that the ADM scalar constraint can be obtained through the following steps. The term linear in the momentum needs to be removed somehow to even match the terms in the ADM scalar constraint. Assuming this is done if the action were re-written by truncating to the first few orders in the derivative expansions shown, or in other words, the constants multiplying the higher-order terms are all set to zero. Further, if the relevant constants are tuned take the values $\lambda = 1 = \gamma = c_1$, $c_0 = \Lambda_{cc}$,[3] then the bulk Hamiltonian will be a sum of the Hamiltonian and diffeomorphism constraints of general relativity. *i.e.*

$$H(\pi_{\mu\nu}, g^{\mu\nu}) = \underbrace{\sqrt{g}(-\Lambda_{cc} + R) + \frac{1}{\sqrt{g}} G^{\mu\nu\alpha\beta}_{\lambda=1} \pi_{\mu\nu}\pi_{\alpha\beta}}_{H_{ADM}(\pi_{\mu\nu}, g^{\mu\nu})} + \cdots . \tag{2.32}$$

The diffeomorphism constraint is the same as that of general relativity. Note that including any of the terms with spatial derivatives of a higher order than those included above in the Hamiltonian will lead to a breakdown of the general covariance of the theory. The reason for this breakdown lies in the mismatch between the number of radial gradients hiding in the two powers of the momenta in the kinetic term, and the spatial gradients. One simple way to see why the ADM Hamiltonian has just the right number of these derivatives is to perform the Legendre transform to find that the Lagrangian thus obtained can be written as a scalar density of weight one formed from the spacetime metric ([11]).

There isn't any reason *a priori* to believe that the aforementioned truncations follow from any of the limits already imposed on the matrix field theory. What would be more satisfying would be to find some additional criteria that the coarse-graining mechanism needs to satisfy from which the ADM form for the constraints follows.

2.4 Wess–Zumino consistency condition

The geometrization of the renormalization group lies in identifying the radial evolution of a constant 'RG time' hypersurface into the bulk with the local quantum renormalization group flow of the boundary theory. This is only strictly true, however, if the evolution is generated by normal deformations of this hypersurface which satisfy a certain commutator or Poisson bracket algebra. This algebra can be seen as a consistency condition for the Hamiltonian evolution because satisfying this condition is necessary for the Hamiltonian flow to not stray away from the subspace of phase space where the constraints are satisfied. And through QRG, it will also reflect the consistency of the local renormalization group flow, in that it dictates how LRG transformations are composed consistently.

[3]Here, Λ_{cc} denotes the Cosmological constant, and shouldn't be confused with the UV cut-off Λ.

The specific form of the structure functions of this algebra is dictated by the diffeomorphism invariance of the $D+1$ dimensional target space into which this hypersurface is embedded. In other words, the property of the generators of the deformations of hypersurfaces forming a certain commutator or Poisson bracket algebra is a sign that these hypersurfaces are embedded into a one higher dimensional (here Euclidean) spacetime. This algebra is known as the hypersurface deformation algebra or the Dirac algebra. This is because it mirrors the Lie bracket algebra of components of spacetime vector fields decomposed tangentially and orthogonally to an embedded hypersurface [148].

In this section, the consequences of imposing the Dirac algebra through the Wess–Zumino consistency condition for the Quantum renormalization group will be investigated. From the field theory perspective, this will amount to relating the anomalous Ward identity for broken Weyl/scale invariance to the Ward identity corresponding to diffeomorphism invariance of the boundary theory coupled its background metric at every scale. This relation between said Ward Identities will then impose restrictions on the algebraic form of the RG Hamiltonian, which is nothing but the Hamiltonian of the dual gravity theory and constrain it to be that of general relativity.

The guarantee that the only representation of this algebra on the gravitational phase space being the ADM constraints follows from a theorem of Hojman, Kuchar, and Teitelboim [138]. Furthermore, the gradient formula for the metric beta function can also be shown to follow from this demand that the constraint algebra close in a specific form. The key result of interest form which this fact shall follow was first proven by Kuchar in [101].

2.4.1 The Wess–Zumino consistency condition as the holographic dual to the Hypersurface deformation algebra

The Wess–Zumino consistency condition for the local renormalization group is simply a statement of the fact that Weyl transformations commute. This means that when two local RG transformations are composed, it doesn't matter which of these transformations are performed first, and which is performed second. Consider the the generating functional and focus on its dependence on the metric: $W[g_{\mu\nu}] \equiv \ln Z[g_{\mu\nu}]$. The statement of the commutativity of the local RG transformations reads

$$[\Delta_{\alpha(x)\delta z}, \Delta_{\alpha'(x)\delta z}] W[g_{\mu\nu}(x)] = 0. \tag{2.33}$$

Even for a conformal field theory on a curved background, this condition imposes non trivial constraints on the form of the conformal anomaly:

$$[\Delta_{\sigma(x)}, \Delta_{\sigma'(x)}] W_{CFT}[g_{\mu\nu}] = \Delta_{\sigma}\mathcal{A}_{\sigma'} - \Delta_{\sigma'}\mathcal{A}_{\sigma} = 0. \tag{2.34}$$

Away from the fixed point, when conformal invariance is broken, these consistency conditions necessarily also lead to non trivial relations the beta functions needs to satisfy, in addition to the anomaly terms.

In the QRG, the beta functions are coded into terms in the Hamiltonian, so the non trivial relations the consistency conditions impose on the beta functions will thus translate into restrictions on the form of the Hamiltonian. In order to see this more concretely, the evaluation of the left hand side of the Wess–Zumino conditions in the QRG context reads:

$$[\Delta_\sigma, \Delta_{\sigma'}]W[g_{\mu\nu}] = \langle [H(\sigma), H(\sigma')] \rangle \underset{N\to\infty}{\to} \langle \{H(\sigma), H(\sigma')\} \rangle, \tag{2.35}$$

the vanishing of this, is how the Wess–Zumino consistency conditions are encoded in the QRG. The diffeomorphism Ward identity does however allow for the possibility that the right-hand side of the action of the commutator of the generators of Weyl transformations to vanish as a consequence of being proportional to the covariant divergence of the energy-momentum tensor. From the QRG perspective, this means that the right-hand side of the bracket between $H(\sigma)$ and itself (smeared with a different lapse multiplier) can in principle be proportional to the constraint H_μ with some smearing perhaps containing the derivatives of the lapse multipliers. This means that the Poisson algebra of the constraints, particularly a specific form of said algebra is the holographic dual to the Wess–Zumino consistency conditions.

We conjecture that the anomalous Ward identity corresponding to the broken Weyl or scale invariance of the theory which the Wess–Zumino consistency conditions pertain is, in a specific way, related to the Ward identity corresponding to the diffeomorphism invariance of the theory.[4] This means that the relationship between these Ward identities imply a specific form of the Poisson algebra of the corresponding dual constraints. This form of the Poisson algebra, given other assumptions we will further mention, will be sufficiently strong to fix the algebraic form of the scalar Hamiltonian constraint to be identical to that of general relativity.

Kinematics of Hypersurface Deformations

To start, it will help to describe the hypersurface deformation algebra at the kinematical level. Consider an infinitesimal spacetime diffeomorphism generated by the vector field

[4]The mathematical statement of which is the covariant conservation of the energy momentum tensor's vacuum expectation value.

v^A[5], *i.e.* $y^A \to y^A + v^A$, it can be decomposed into components tangential and orthogonal to any given hypersurface as

$$v^A = \sigma n^A + v_\parallel^A, \tag{2.36}$$

where $\sigma = n^A v_A$, and $v_\parallel^A = -(n^C v_C)n^A + v^A$. Note that this vector is purely tangential to the hypersurface, because $n_A v_\parallel^A = 0$, so we will denote it as $v_\parallel^\mu$ in what follows. The vector n^A is the normal to a co-dimension one hypersurface Σ. The deformation of the hypersurface generated by the vector field v^A is given through the action of the operator

$$X(v) = \int_\Sigma \mathrm{d}^D x \sqrt{g} v^A \frac{\delta}{\delta y^A}, \tag{2.37}$$

which satisfies the commutation relations

$$[X(v), X(w)] = X([v, w]). \tag{2.38}$$

Here $[v, w]$ is the Lie bracket of the vector fields v^A, w^A. Then a foliation dependent decomposition of the above operator can be introduced as follows:

$$N_\sigma = \int_\Sigma \mathrm{d}^D x \sqrt{g} \sigma n^A \frac{\delta}{\delta y^A}, \tag{2.39}$$

$$T_{v_\parallel} = \int_\Sigma \mathrm{d}^D x \sqrt{g} v_\parallel^\mu \partial_\mu y^A \frac{\delta}{\delta y^A}. \tag{2.40}$$

The algebra of these deformations is given by

$$[N_\sigma, N_{\sigma'}] = -T_{f(\sigma,\sigma')}, \quad [T_{v_\parallel}, N_\sigma] = -N_{v_\parallel^\mu \partial_\mu \sigma}, \quad [T_{v_\parallel}, T_{w_\parallel}] = T_{[v_\parallel, w_\parallel]}. \tag{2.41}$$

Here $f^\nu(\sigma, \sigma') = g^{\mu\nu}(\sigma' \partial_\mu \sigma - \sigma \partial_\mu \sigma')$.

It is interesting to see that the above Lie bracket algebra is not a Lie algebra because the analog of the structure constants is now replaced by phase space-dependent functions, *i.e.* the vector $f^\nu(\sigma, \sigma')$. It is still analogous to a Lie algebra in the sense that the commutator of these deformations generators closes to other deformation generators. Also, the structure functions of the above deformation algebra are fixed by the demand that when the normal and tangential deformations are combined to form the overall deformation $X(v)$, it satisfied the algebra of spacetime diffeomorphisms (2.37). In other words, the specific form of the structure functions of this algebra is fixed by the demand for full diffeomorphism invariance of the spacetime into which the hypersurface is embedded. This algebra must be mirrored by the Poisson algebra of the constraints on the phase space of any dynamical theory which respects full diffeomorphism invariance. This fact that spacetime structure is reflected in the algebra of constraints was described in [148].

[5] we will use uppercase latin letters such as $A, B, ...$ for $D+1$ dimensional spacetime tensors, which will run form 0 to D.

Implications for gravity

The result of key importance in the context of the phase space of general relativity is that of Hojman, Kuchar and Teitelboim (HKT) in [138]. They prove that the unique representation of the algebra of hypersurface deformations (2.41) on the phase space spanned by the metric on a hypersurface and its conjugate momentum is given by the following constraints:

$$N_\sigma \to \int_\Sigma \mathrm{d}^D x \sigma(x) \left(\frac{1}{\sqrt{g}} \left(\pi_{\mu\nu}\pi^{\mu\nu} - \frac{1}{D-1}\mathrm{tr}\pi^2 \right) - \sqrt{g}(\Lambda_{cc} - R) \right),$$

$$T_{v_\parallel} \to \int_\Sigma \mathrm{d}^D x \sqrt{g} v_\parallel^\nu(x)(\nabla_\mu \pi^\mu_\nu). \tag{2.42}$$

The Poisson algebra of these constraints mirrors the algebra of hypersurface deformations. These constraint functions are easily recognised as the ADM scalar and diffeomorphism constraints where the lapse Lagrange multiplier is identified with $\sigma(x)$ and the shift multiplier is identified with $v_\parallel^\mu(x)$. Thus the following Poisson algebra being satisfied by the constraints:

$$\{H(\sigma), H(\sigma')\} = -H_\mu(f^\mu(\sigma,\sigma')),\ \{H_\mu(\xi^\mu), H(\sigma)\} = H(\xi^\mu \partial_\mu \sigma),$$

$$\{H_\mu(\xi^\mu), H_\nu(\zeta^\nu)\} = H_\mu([\xi,\zeta]^\mu). \tag{2.43}$$

is a necessary and sufficient condition for these constraints to take the ADM form.

The third of the above Poisson bracket relations is a representation of the algebra of spatial diffeomorphism algebra. The second bracket is entirely a consequence of the fact that the scalar constraint density is a tensor density of weight one. In a sense, these brackets pertain to just kinematics, as far as QRG is concerned. This is because the diffeomorphism constraint in the total QRG Hamiltonian is already of the same algebraic form as the diffeomorphism constraint of GR, and hence necessarily satisfies the same Poisson algebra. Also, the tentative scalar constraint density being a tensor density of weight one only instructs where factors of $\sqrt{g}$ ought to appear in each of its terms, but it doesn't fix the functional dependence of the functions themselves on the phase space variables. The first Poisson bracket relation however does indeed pertain to dynamics. For it to be satisfied, the form of various functions in the scalar constraint is fixed.

Going back to the quantum renormalization group, there is now potential to impose a condition on the very coarse graining scheme itself, the satisfaction of which will force the RG Hamiltonian to take the ADM form. This condition is the holographic dual to the hypersurface deformation algebra. It is given by:

$$[\Delta_\sigma, \Delta_{\sigma'}]W[g_{\mu\nu}] = \int \mathrm{d}^3x\sqrt{g}f^\mu(\sigma,\sigma')\langle\nabla_\rho T^\rho_\mu\rangle, \tag{2.44}$$

which, as mentioned before is but a particular manner in which the Wess Zumino consistency condition is satisfied (because $\langle\nabla_\rho T^\rho_\mu\rangle = 0$).

We will now describe how the functions in the QRG Hamiltonian can be fixed by demanding this particular form of the consistency condition, starting with the kinetic term.

2.4.2 The Kinetic term

The tentative kinetic term is one which is quadratic in the momentum, but only the first term in the gradient expansion is ultra local in the metric and momenta. The QRG scalar constraint in this case takes the form

$$H(\sigma) = \int_\Sigma \mathrm{d}^Dx\alpha(x,z)\left(\frac{1}{\sqrt{g}}\left(\pi_{\mu\nu}\pi^{\mu\nu} - \frac{1}{D-1}\mathrm{tr}\pi^2\right) + F(g,\pi) - \beta_{\mu\nu}(g)\pi^{\mu\nu} + \sqrt{g}V(g)\right). \tag{2.45}$$

The function $F(g,\pi)$ stands for the rest of the terms in the gradient expansion of the quadratic in momentum term. This is a function which consists of an arbitrary number of derivatives of the metric and the two powers of momenta. The key Poisson bracket relation to use in order to fix the form of the remaining functions in this constraint is the bracket between two scalar constraints, *i.e.*

$$\{H(\alpha), H(\alpha')\} = H_\mu(f^\mu(\alpha,\alpha')), \tag{2.46}$$

where leaving aside the specific form of $f^\mu(\alpha,\alpha')$, a lot can be gained from just noticing the fact that the right hand side of the above bracket is linear in the momentum. In order to exploit this feature, it is useful first to recall that the Poisson brackets between two phase space functions reduce the total polynomial order in the momenta by one and maintain the order of spatial derivatives. The bracket between two scalar constraints breaks up into a sum of several terms which can be ordered based on the total polynomial order of the momenta. The highest order term from this counting would be the bracket between the kinetic term and itself:

$$\left\{\int_\Sigma \mathrm{d}^Dx\alpha(x,z)\left(\frac{1}{\sqrt{g}}G^{\mu\nu\alpha\beta}\pi_{\mu\nu}\pi_{\alpha\beta} + F(g,\pi)\right), \int_\Sigma \mathrm{d}^Dy\alpha'(y,z)\left(\frac{1}{\sqrt{g}}G^{\mu\nu\alpha\beta}\pi_{\mu\nu}\pi_{\alpha\beta} + F(g,\pi)\right)\right\}$$

$$-(\alpha \leftrightarrow \alpha'), \tag{2.47}$$

This will split up again into three terms, the first being

$$\left\{ \int_\Sigma \mathrm{d}^D x \frac{\alpha(x,z)}{\sqrt{g}} G^{\mu\nu\alpha\beta}\pi_{\mu\nu}\pi_{\alpha\beta}, \int_\Sigma \mathrm{d}^D y \frac{\alpha'(y,z)}{\sqrt{g}} G^{\mu\nu\alpha\beta}\pi_{\mu\nu}\pi_{\alpha\beta} \right\} - (\alpha \leftrightarrow \alpha') \tag{2.48}$$

which vanishes because of the ultra locality of the resulting expression and subsequent anti-symmetrisation of the smearing functions. The second and third terms of the form

$$\left\{ \int_\Sigma \mathrm{d}^D x \frac{\alpha(x,z)}{\sqrt{g}} G^{\mu\nu\alpha\beta}\pi_{\mu\nu}\pi_{\alpha\beta}, \int_\Sigma \mathrm{d}^D y \alpha'(y,z) F(g,\pi) \right\} - (\alpha \leftrightarrow \alpha'), \tag{2.49}$$

$$\left\{ \int_\Sigma \mathrm{d}^D x \alpha(x,z) F(g,\pi), \int_\Sigma \mathrm{d}^D y \alpha'(y,z) F(g,\pi) \right\} - (\alpha \leftrightarrow \alpha'), \tag{2.50}$$

don't however identically vanish due to the presence of spatial gradients, and thus leads to a set of terms genuinely cubic in the momenta. These terms vanish if and only if $F(g,\pi) = 0$. The study of such terms is the subject of [140]. In that work, the demand that the structure functions be independent of the momenta even is not imposed, but still, the constraint algebra doesn't remain first class under a wide class of such modifications of the kinetic term. This eliminates the potentially cubic term in the result of this Poisson bracket relation.

Thus from just positing this holographic version of the Wess–Zumino consistency condition, it follows that

$$G^{\mu\nu\alpha\beta\{\eta\}\{\rho\}} \to \frac{1}{\sqrt{g}} G^{\mu\nu\alpha\beta} = \frac{1}{\sqrt{g}} (g^{\mu(\alpha} g^{\beta)\nu} - g^{\mu\nu} g^{\alpha\beta}). \tag{2.51}$$

Thus the double trace beta function is an ultra local function of the metric known as the de-Witt super metric. This also implies that the kinetic term of the Hamiltonian is ultra local in both the metric and the momenta. This is a canonically normalised kinetic term, akin to that which is encountered in most field theories.

The super metric is a metric on the space of metrics is used to define an inner product on field space, which in turn is necessary to define the functional integral over geometries in the quantum theory. Paraphrasing from a discussion in [119], allowing this inner product to be taken with respect to a super metric containing derivatives of the metric would have the effect of defining a different set of dynamical fields in the theory. So the specification of an ultra-local super metric can also be seen as a manifestation of the fact that the metric is taken to be the fundamental field variable and that the presence of derivatives of it in the action is the cause for dynamics.

Furthermore, the ultra locality of the kinetic term in both the metric and the momenta will ensure the invertibility of the relationship between the canonical momenta and the extrinsic curvature tensor. The extrinsic curvature tensor is defined as

$$K_{\mu\nu} = -\frac{1}{2}\mathcal{L}_n g_{\mu\nu}, \tag{2.52}$$

where n^μ is the vector normal to the hypersurface. The relationship between this tensor and the canonical momentum given the ultra-local kinetic term is

$$K_{\mu\nu} = \frac{1}{\sqrt{g}}\left(\pi_{\mu\nu} - \frac{1}{D-1}\mathrm{tr}\pi g_{\mu\nu}\right). \tag{2.53}$$

The simple algebraic nature of the relationship between the canonical momenta and the extrinsic curvature is one of the many necessary conditions to find a Lagrangian that can be rewritten in the form of the Einstein Hilbert action which is manifestly covariant in $D+1$ dimensions after performing the Legendre transform.

The HKT result also seems to demand that the metric beta function ought to simply vanish for the algebra of constraints to be satisfied, and also, the potential term should be truncated to just the first two terms in its derivative expansion. There is a subtlety here regarding the fate of the term linear in the momentum and the potential term and it will be elucidated and addressed in the subsection to follow.

2.4.3 Gradient flow formula for the metric beta function and canonical transformations

The constraint algebra enforced through the holographic Wess–Zumino consistency conditions also has implications for the form of the beta function and potential term in the RG Hamiltonian.

In the last section, it was deduced that the kinetic term should be ultra-local in both the metric and the momenta to satisfy the hypersurface deformation algebra. Assuming only this, the scalar constraint is given by

$$H(\alpha) = \int_\Sigma \mathrm{d}^D x\alpha(x,z)\left(\frac{1}{\sqrt{g}}\left(\pi_{\mu\nu}\pi^{\mu\nu} - \frac{1}{D-1}\mathrm{tr}\pi^2\right) - \beta_{\mu\nu}(g)\pi^{\mu\nu} + \sqrt{g}V(g)\right). \tag{2.54}$$

We will now sketch the derivation of the result showing that the demand that the hypersurface deformation algebra is satisfied will translate into the so-called 'gradient formula' for the metric beta function. For more detailed computations proceeding along this line of reasoning to derive this result, see [140]. The original derivation of this result came from an effort to formulate the HKT theorem in the Lagrangian framework by Kuchar in [101].

Given that the vanishing of the cubic resulting from the Poisson brackets has been established, the next to higher order term will be a quadratic in momentum expression which comes from the bracket between the kinetic term and the term linear in the momentum:

$$\left\{ \int_\Sigma \mathrm{d}^D x \frac{\alpha(x,z)}{\sqrt{g}} G^{\mu\nu\alpha\beta}\pi_{\mu\nu}\pi_{\alpha\beta}, \int_\Sigma \mathrm{d}^D y \alpha'(y,z)\beta_{\mu\nu}(g)\pi^{\mu\nu} \right\} - (\alpha \leftrightarrow \alpha'). \tag{2.55}$$

The function $\beta^{\mu\nu}(g)$ depends on the metric and its momenta, and the above expression will not identically vanish despite the anti-symmetrisation of the smearing functions. In order to satisfy the hypersurface deformation algebra however, this expression must strongly vanish, by virtue of the fact that there is no term on the right hand side of (2.46) that is quadratic in the momenta.

It can be shown [6] that the vanishing of the above quadratic in momentum term will imply that

$$\frac{\delta(G^{\alpha\gamma\mu\nu}\beta_{\alpha\gamma})(x)}{\delta g_{\rho\eta}(y)} - \frac{\delta(G^{\delta\kappa\rho\eta}\beta_{\delta\kappa})(y)}{\delta g_{\mu\nu}(x)} = 0. \tag{2.56}$$

Following similar logic, a term generated by the Poisson bracket calculation that is momentum independent should also vanish. The relevant piece of the bracket here will be

$$\left\{ \int_\Sigma \mathrm{d}^D x \alpha(x,z)\beta_{\mu\nu}(g)\pi^{\mu\nu}, \int_\Sigma \mathrm{d}^D y \sqrt{g}\alpha'(y,z)V(g) \right\} - (\alpha \leftrightarrow \alpha'), \tag{2.57}$$

which produces a term that is independent of the momenta and can be split up into a sum of terms ordered by the number of spatial gradients. The first non-trivial order of derivatives will be the second order, the vanishing of which is

$$\nabla_\gamma(G^{\alpha\gamma\mu\nu}\beta_{\mu\nu}) = 0. \tag{2.58}$$

The conditions (2.56) and (2.58) then imply that the function $\beta_{\mu\nu}(g)$[7] has to take the form

$$\beta_{\mu\nu}(g) = G_{\mu\nu\rho\eta}\frac{\delta c[g]}{\delta g_{\rho\eta}}, \tag{2.59}$$

[6] See section 5 of [140]

[7] for a sketch of the proof of this statement, see [101] and references therein

where $c[g]$ is a functional of the metric and its derivatives.

From the quantum field theory perspective, the equation (2.59) is the so-called gradient formula for the metric beta function. This result was arrived at solely through considerations of the ('holographic') Wess–Zumino consistency conditions much akin to how such a formula is derived in the traditional local RG literature, for instance in [?], [?]. Such a formula was also derived from considerations of entanglement entropy in holographic theories in [94], in the case where $c[g]$ takes the form of the Einstein Hilbert action.

Coming back to the dual gravitational theory, consider just the kinetic term and the term linear in the momentum, the sum of which can be manipulated as follows

$$\begin{aligned}&\frac{1}{\sqrt{g}}G_{\mu\nu\rho\eta}\pi^{\mu\nu}\pi^{\rho\eta}-G_{\mu\nu\rho\eta}\frac{\delta c[g]}{\delta g_{\rho\eta}}\pi^{\mu\nu}\\ &=\frac{1}{\sqrt{g}}G_{\mu\nu\rho\eta}\left(\pi^{\mu\nu}-\frac{1}{2}\frac{\delta c[g]}{\delta g_{\mu\nu}}\right)\left(\pi^{\rho\eta}-\frac{1}{2}\frac{\delta c[g]}{\delta g_{\rho\eta}}\right)-\frac{1}{4}G_{\mu\nu\rho\eta}\frac{\delta c[g]}{\delta g_{\mu\nu}}\frac{\delta c[g]}{\delta g_{\rho\eta}}.\end{aligned} \tag{2.60}$$

This manipulation makes the possibility for the following canonical transformation

$$\pi^{\mu\nu}\to\pi^{\mu\nu}-\frac{1}{2}\frac{\delta c[g]}{\delta g_{\mu\nu}},$$

apparent. This is a canonical transformation because it preserves the canonical Poisson brackets of the theory and subsequently comes at the cost of adding a total derivative term to the action. The role of such canonical transformations in holographic RG was discussed in detail in [128]. It follows from this canonical transformation that

$$\begin{aligned}\int\mathrm{d}z\int_\Sigma\mathrm{d}^Dx\sqrt{g}\pi^{\mu\nu}\dot{g}_{\mu\nu}&\to\int\mathrm{d}z\int_\Sigma\mathrm{d}^Dx\sqrt{g}\pi^{\mu\nu}\dot{g}_{\mu\nu}+\int\mathrm{d}z\int_\Sigma\mathrm{d}^Dx\sqrt{g}\frac{1}{2}\frac{\delta c[g]}{\delta g_{\mu\nu}}\dot{g}_{\mu\nu}\\ &=>\int\mathrm{d}z\int_\Sigma\mathrm{d}^Dx\sqrt{g}\pi^{\mu\nu}\dot{g}_{\mu\nu}+c[g]|_{z=0}^{z=z_*}.\end{aligned} \tag{2.61}$$

This is just the statement of the fact that $c[g]$ is the generating functional of the aforementioned canonical transformation.

This effectively removes the linear term in the scalar constraint, leaving only the ultra-local, canonical kinetic term. The last term in the Hamiltonian constraint whose form hasn't yet been fixed in the above discussion from the demand of satisfaction of the hypersurface deformation algebra is the momentum independent potential term. It too gets modified as a consequence of the canonical transformation mentioned above, *i.e.*

$$V(g)\to V(g)-\frac{1}{4}G_{\mu\nu\rho\eta}\beta^{\mu\nu}(g)\beta^{\rho\eta}(g)\equiv U(g). \tag{2.62}$$

Now, the form of the Hamiltonian constraint density after the canonical transformation is:

$$\frac{1}{\sqrt{g}} G_{\mu\nu\rho\eta} \pi^{\mu\nu} \pi^{\rho\eta} + U(g),$$

and the HKT result will lead to the condition

$$U(g) = V(g) - \frac{1}{4} G_{\mu\nu\rho\eta} \beta^{\mu\nu}(g) \beta^{\rho\eta}(g) = \sqrt{g}(-\Lambda_{cc} + R). \tag{2.63}$$

This difference between the potential term and the square of the beta function being exactly the potential term in the ADM Hamiltonian constraint is also related to the vanishing of the difference between the a and c anomaly coefficients in AdS/CFT as was discussed in [123].[8]

We wish to emphasize that the above subsection provides a *derivation* of the gradient condition for the metric beta function which was so far assumed in discussions relating to the quantum renormalization group. The additional input however was the holographic Wess–Zumino consistency conditions. On the gravity side of the duality, these conditions translate into the closure of the constraint algebra in a very specific manner, which is a stronger condition than just the demand for closure of the constraint algebra which was already presented in [102].

2.4.4 The realm of possibilities

Despite the many arguments made in the previous sections to justify the conjectured form of the Wess–Zumino consistency conditions, one can nevertheless ask what other consistent choices could have been made on the gravity side for these conditions to be satisfied. A consistent choice of how the Wess–Zumino conditions are satisfied translates through the duality into a manner in which the Poisson algebra of constraints can close. Then, the most general condition one can impose on the bracket between two scalar constraints is just closure *i.e.* to require

$$\{H(\sigma), H(\sigma')\} \approx 0. \tag{2.64}$$

The symbol $\approx$ denotes "weak equality" which means equality when the constraints are satisfied. That would leave the possibility for the Poisson brackets to result in terms proportional to both the scalar and vector constraint with arbitrary structure functions, whose phase space dependence is made explicit with the notation $\tilde{f}^\mu(g, \pi; \sigma, \sigma')$, $\tilde{h}(g, \pi; \sigma, \sigma')$:

$$\{H(\sigma), H(\sigma')\} = H(\tilde{h}(g, \pi; \sigma, \sigma')) + H_\mu(\tilde{f}^\mu(g, \pi; \sigma, \sigma')). \tag{2.65}$$

[8] It should be noted that conventions to do with factors of 2 differ between this chapter and [123].

Before proceeding further in this discussion, it will help to first take a step back and recall some basic notions in the theory of constrained Hamiltonian systems. The fact that the Poisson algebra of the constraints results in terms proportional to the constraints themselves implies that the constraint algebra is *first class*. The first-class nature of the $D+1$ constraints in the general relativity context is the manifestation of the fact that there are $\frac{(D+1)(D-2)}{2}$ true degrees of freedom of the gravitational field.

Now, going back to the situation of interest, one can ask what class of gravitational theories (i.e. theories defined on the phase space of GR) exist that possess spatial (*i.e.* in the field theory's space directions) diffeomorphism invariance and a local quadratic in momentum Hamiltonian constraint which is first class and hence propagate the same number of degrees of freedom as general relativity. No such theory has been found so far, although a complete proof of the statement that no such theory could be found doesn't exist at the moment. Nevertheless, if additional restrictions such as the demand that the kinetic term is ultra-local are imposed, then the mere demand for closure of the constraint algebra will force the 'tentative' constraints to take the form of those of general relativity, as was shown in [56]. This remains true if the kinetic term is also modified by the addition of an arbitrary local, but quadratic in momentum term, see [70]. If the demand that the modifications no longer remain quadratic in momenta is relaxed, then perhaps there is a wide range of generalizations of the hypersurface deformation algebra that are admissible, such as those described in [21].

If no such theory exists, then what one would conclude is that the *only* realization of a first-class scalar constraint in otherwise spatially covariant theory of gravity is necessarily the ADM Hamiltonian constraint of general relativity. In that case, the only demand that one need impose is for the Wess–Zumino conditions to *somehow* be satisfied, *i.e.* that the algebra of constraint simply be first class and the only consistent manner in which such closure can be achieved is if the constraints are those of general relativity. Here, there will be no need to make any conjecture about the specific manner in which the Wess–Zumino consistency conditions are satisfied, and the covariance of the dual theory will follow solely from the abelian nature of the group of local Weyl transformations.

2.5 Holographic Renormalization

The previous sections dealt with aspects of the renormalization group flow which in principle will hold even if the theory under consideration is an effective field theory with a finite UV cutoff. This means that any solution to the flow equations cannot be extended to infinite flow time. The step of renormalization is to find a set of flows emanating from an ultraviolet fixed point so that the flow time can be taken to infinity. Corresponding to that there is a chart on the theory space with well-defined transition maps corresponding to the renormalized sources. This perspective is emphasized in [108].

To define renormalized correlation functions and other such observables, one need only renormalize the generating functional. In the Hamiltonian system the flow gets mapped to, this corresponds to finding the on-shell action defined with boundary conditions at large flow times. The on-shell action is equal to Hamilton's principle functional, so to study renormalization of the boundary theory, we need to solve the Hamilton–Jacobi equation with large (radial) time asymptotics. This procedure is known as holographic renormalization (see for instance [40]). The approach followed henceforth is that of Skenderis, Papadimitriou et. al. (see for instance[128], [130], [131] and references therein) and is known as the Hamiltonian approach to holographic renormalization. First, it will help to clarify the role of the dilatation operator.

2.5.1 The Dilatation Operator

To describe a theory at the UV fixed point, one need recover the conformal Ward identities at infinite flow time (from the perspective of the Hamiltonian system). As $\tau \to \infty$, we need

$$\partial_\tau \sim \int_x 2g^{\mu\nu}\frac{\delta}{\delta g^{\mu\nu}} \equiv \delta_{\sigma(x)=1},$$

for later convenience, we denote $\delta_{\sigma(x)=1} = \delta_g$

The Hamiltonian approach to Holographic renormalization is based on expanding Hamilton's principal functional in eigenvalues of this operator.

This identification corresponds to an asymptotic fixing of the Lagrange multipliers to $(\sigma, \xi^\mu) = (1, 0)$. Also, from the relation

$$\dot{g}_{\mu\nu} \overset{\tau\to\infty}{\sim} \delta_g g_{\mu\nu} = 2g_{\mu\nu}, \tag{2.66}$$

one can read off the asymptotic limit of the metric $g_{\mu\nu}|_{\tau\to\infty} = e^{2\tau}g_{(0)\mu\nu}$. So we see that the action of the Hamiltonian and that of the dilatation operator at infinity is thus identified simply.

This type of behavior hints at asymptotic boundary conditions of the Anti-de Sitter type (assuming the radial direction is Euclidean). To see clearly why the holographic UV conformal field theory is dual to asymptotically AdS spacetime, will illustrate how to solve the Hamilton–Jacobi equations, but in summary, it has to do with the dilatation being identified with radial evolution at infinity. Therefore, the holographic counterpart to the renormalization procedure is to characterize asymptotic infinity which is seen as the limit as the radial coordinate is taken to infinity. The construction of the requisite counterterms needed to define the on-shell action thus coincide with those required to renormalize the dual quantum field theory, as both the infrared divergence in the former as well as the ultraviolet divergence in the latter arise in the $\tau \to \infty$ limit.

Now we will demonstrate more explicitly how the conformal algebra emerges at the fixed point

2.5.2 The Conformal Algebra

In the previous section, we see that when considering the U.V. fixed point of the theory the background metric takes the form $g_{\mu\nu} = e^{2\tau} g_{(0)\mu\nu}$ where $\tau \to \infty$. In this section, we will utilise this and focus on the case where $g_{(0)\mu\nu} = \eta_{\mu\nu}$, i.e. when the metric is conformally flat. Also, if we choose the the diffeomorphism transformations to be generated by

$$\xi^\mu = a^\mu + \omega^\mu_\nu x^\nu + \lambda x^\mu + 2(b \cdot x)x^\mu - x^2 b^\mu, \tag{2.67}$$

and the Weyl factor given by $\frac{\partial \cdot \xi}{D}$,

$$\sigma(x) = \frac{\partial \cdot \xi}{D} = \lambda + 2b \cdot x. \tag{2.68}$$

It then follows that the local RG transformations and diffeomorphisms generated by

$$H(\lambda) + H_\mu(\lambda x^\mu) = D^{(\lambda)},\ H(2b \cdot x) + H(2b \cdot x x^\mu - x^2 b^\mu) = K^{(b)},$$

$$H_\mu(a^\mu) = P^{(a)},\ H_\mu(\omega^\mu_\nu x^\nu) = J^{(\omega)}.$$

Then, it follows from the constraint algebra that these generators satisfy the following algebra:

$$\left\{K^{(b)}, D^{(\lambda)}\right\} = -K^{(\lambda b)},\ \left\{P^{(a)}, D^{(\lambda)}\right\} = P^{(\lambda a)},\ \left\{K^{(b)}, P^{(a)}\right\} = -D^{(a \cdot b)} + J^{(a \times b)}, \tag{2.69}$$

$$\left\{J^{(\omega)}, K^{(b)}\right\} = K^{(\omega \cdot b)},\ \left\{J^{(\omega)}, P^{(a)}\right\} = P^{(\omega \cdot a)},\ \left\{J^{(\omega)}, J^{(\omega')}\right\} = J^{(\omega \cdot \omega')}.$$

Here $a \times b \equiv a^{[\mu} b^{\nu]}$, $\omega \cdot a \equiv \omega^\mu_\nu a^\nu$, $\omega \cdot b \equiv \omega^\mu_\nu b^\nu$ and $\omega \cdot \omega' \equiv \omega^\mu_\nu \omega'^\nu_\rho$. This is the conformal algebra that emerges as the residual symmetry group of the background geometry that arises at the UV fixed point. Note that in order to derive the above algebra, it is crucial that to note that the background given by the conformally flat metric with the infinite Weyl factor, the Hamiltonian constraint commutes strongly. This is because the structure function vanishes $e^{-2\tau}g^{\mu\nu}_{(0)}(\sigma\partial_\nu\sigma' - \sigma'\partial_\nu\sigma) \to 0$ as $\tau \to \infty$. This is consistent with the fact that Weyl transformations commute.

2.5.3 Hamilton–Jacobi Equations

The Hamilton–Jacobi equation is obtained from setting the momenta equal to derivatives of Hamilton's principal functional:

$$\pi^{\mu\nu} = \frac{\delta \mathcal{S}}{\delta g_{\mu\nu}} \tag{2.70}$$

This is nothing but a restatement of (2.3) making the identification of the generating functional with Hamilton's principal functional. The Hamilton–Jacobi equations then take the form

$$G_{\mu\nu\rho\sigma}\frac{\delta \mathcal{S}}{\delta g_{\mu\nu}}\frac{\delta \mathcal{S}}{\delta g_{\rho\sigma}} - \sqrt{g}(R - 2\Lambda) = 0, \tag{2.71}$$

$$\nabla_\nu \frac{\delta \mathcal{S}}{\delta g_{\mu\nu}} = 0 \tag{2.72}$$

These are nothing but a restatement of the local renormalization group equations satisfying the holographic Wess–Zumino consistency conditions.

The Hamilton–Jacobi equations can also be seen as a canonical transformation away from zero momenta. From this perspective, the beta functions are given by the gradient formula where $\mathcal{C}$ is replaced by $\mathcal{S}$. To study the fixed point, it will suffice to consider pure gravity, as all other beta functions would vanish here.

2.5.4 Solving the Hamilton–Jacobi Equations

The approach followed in this subsection and the next summarises what is described in detail in [128], [130], and introduced in [131].

The Hamiltonian approach to Holographic renormalization is based on expanding Hamilton's principal functional in eigenfunctions of δ_g:

$$\mathcal{S} = \mathcal{S}_{(0)} + \mathcal{S}_{(1)} + \mathcal{S}_{(2)} + \cdots\ ,\ \mathcal{S}_{(k)} = \int \mathrm{d}^D x \sqrt{g}\mathcal{L}_{(k)}. \tag{2.73}$$

These functions satisfy

$$\delta_g \mathcal{S}_{(k)} = (D-k)\mathcal{S}_{(k)}. \tag{2.74}$$

The momenta thus also admit an expansion of the form

$$\pi^{\mu\nu} = \pi^{\mu\nu}_{(0)} + \pi^{\mu\nu}_{(1)} + \pi^{\mu\nu}_{(2)} + \cdots \tag{2.75}$$

So the relation (2.74) can be written through the momenta as

$$\mathrm{tr}\pi_{(k)} = (D-k)\mathcal{L}_{(k)}, \tag{2.76}$$

and this relation will be useful in the discussion to follow. The case of interest is to take $\tau \to \infty$ and pick out the divergent terms to cancel with counter-terms. This relation can be used to iteratively solve the Hamilton–Jacobi equations but this procedure can be followed only up to $\mathcal{S}_{(D)}$ due to poles. Given that the potential counter-terms of interest are all pertinent to operators of dimension D or lower, this expansion will suffice to construct the counterterms and compute the anomaly. These counterterms can be identified as

$$\mathcal{S}_{ct} = -\sum_{k=0}^{D} \mathcal{S}_{(k)}. \tag{2.77}$$

The anomaly stems from the logarithmically divergent terms which appear for example in four dimensions with the pole $\frac{1}{D-4}$, that is converted through dimensional regularisation into $-\ln(e^{-2\tau})/2$. These must be summed together to obtain the logarithmically divergent term $\tilde{\mathcal{L}}_{(D)}$. The renormalized action is defined as

$$\mathcal{S}_{ren} = \lim_{\tau\to\infty}(\mathcal{S} + \mathcal{S}_{ct}) = \lim_{\tau\to\infty}\int \mathrm{d}^D x \sqrt{g}\mathcal{L}_{(D)}. \tag{2.78}$$

To start with, begin with the zeroth order Hamilton–Jacobi equation:

$$G_{\mu\nu\rho\sigma}\frac{\delta\mathcal{S}_{(0)}}{\delta g_{\mu\nu}}\frac{\delta\mathcal{S}_{(0)}}{\delta g_{\rho\sigma}} - \sqrt{g}\Lambda = 0, \tag{2.79}$$

which admits the solution $\mathcal{S}_{(0)} = \int \sqrt{g}c_1$, where $c_1 \propto \sqrt{-\Lambda}$, and in appropriate units of the cosmological length scale, this can be set to one. We will specialise to $D = 4$ for simplicity. To second order, the Hamilton–Jacobi equation reads

$$-\frac{2}{D-1}g_{\mu\nu}\frac{\delta\mathcal{S}_{(2)}}{\delta g_{\mu\nu}} + \sqrt{g}R = 0, \tag{2.80}$$

which admits a solution $\mathcal{S}_{(2)} = \int \mathrm{d}^D x \sqrt{g} c_2 R$, where $c_2 = \frac{(D-2)}{(D-1)}$.

Finally, the fourth order HJE in $D = 4$ reads

$$g_{\mu\nu} \frac{\delta \mathcal{S}_{(4)}}{\delta g_{\mu\nu}} + G_{\mu\nu\rho\sigma} \pi^{\mu\nu}_{(2)} \pi^{\rho\sigma}_{(2)} = 0, \tag{2.81}$$

and from knowing $\mathcal{S}_{(2)}$, we see that the right hands side reads

$$G_{\mu\nu\rho\sigma} \pi^{\mu\nu}_{(2)} \pi^{\rho\sigma}_{(2)} = \sqrt{g} \left(\frac{2}{3} R^2 - 2 R^{\mu\nu} R_{\mu\nu} \right).$$

This leads to the counter term action:

$$\mathcal{S}_{ct} = \int \sqrt{g} \left[1 + 2 \left(R - \ln(e^{-2\tau}) \left(\frac{1}{3} R^2 - R^{\mu\nu} R_{\mu\nu} \right) \right) \right]. \tag{2.82}$$

The fourth order equation has another interpretation as the renormalised trace Ward–Identity.

2.5.5 Renormalized Trace Ward Identity and the Anomaly

Recall the relation

$$\mathrm{tr}\pi_{(4)} = (D-4) \mathcal{L}_{(4)},$$

which as $D = 4$ is ill defined. To cure this, dimensional regularisation is employed, as mentioned before to set $\tau_0 = \frac{1}{D-4}$, and then one defines

$$\mathcal{L}_{(4)}|_{\tau_0} = -2\tau_0 \tilde{\mathcal{L}}_{(4)}|_{\tau_0},$$

so that

$$\tilde{\mathcal{L}}_{(4)} = -\frac{1}{2} \mathrm{tr}\pi_{(4)}. \tag{2.83}$$

This is finite as $\tau \to \infty$ and in that limit this defines the trace Ward identity, again from recalling the relation $\lim_{\tau\to\infty} \mathrm{tr}\pi_{(4)} = \langle T^\mu_\mu \rangle_{ren}$. So, the interpretation is that this is the expression for the anomaly at the fixed point:

$$\langle T^\mu_\mu \rangle_{ren} = \mathcal{A}_h = \left(\frac{1}{3} R^2 - R^{\mu\nu} R_{\mu\nu} \right). \tag{2.84}$$

This is known as the Holographic anomaly, which is one of the celebrated results of the AdS/CFT correspondence: [85], [92]. Comparing this to the general expression for the anomaly in conformal field theories in four dimensions, i.e.

$$\mathcal{A} = \left(\frac{c}{3} - a\right) R^2 + (-2c + 4a) R^{\mu\nu} R_{\mu\nu} + (a - c) R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma},$$

we see that the relation $a = c$ between the anomaly coefficients is implied by the above expression.

From the fact that the asymptotic generator of radial translations is equated with the generator of dilatations, as is expressed in (2.66), we see that the renormalized Hamiltonian should, therefore, be equated with $\mathrm{tr}\pi_{(4)}$ at $\tau \to \infty$. Thus the renormalized Hamiltonian is equal to the holographic anomaly:

$$H_{ren} = -\langle T^{\mu}_{\mu} \rangle_{ren} = \mathcal{A}_h. \tag{2.85}$$

Another way to see how the $a = c$ condition arises in the local holographic RG was demonstrated by Nakayama in [123].

Chapter 3

Flows satisfying the Holographic Wess–Zumino consistency conditions

In the previous chapter, we discussed the Holographic Wess–Zumino consistency condition and how it reflects the commutator or Poisson bracket algebra of the generators of deformations of a hypersurface embedded in a one higher dimensional space time. Since it is a re-writing of the bulk space-time diffeomorphism algebra, the asymptotic conformal symmetry algebra arises from this algebra at the boundary of an asymptotically AdS space. The limit of approaching the boundary of the AdS space mirrors the limit of taking the cutoff to zero in the theory on the boundary. What is missing in this story is and identification of the effective field theory that inhabits the surface at some finite radial slice. Examples of such theories will be the focus of this chapter.

The paradigmatic example will be the $T\bar{T}$ deformation of large c holographic, two dimensional conformal field theories.

3.1 QRG meets $T\bar{T}$ deformed CFT$_2$

In this section, we will discuss applying the QRG procedure to the $T\bar{T}$ flow of large c two-dimensional conformal field theories, and how this leads to general relativity in AdS_3. Most importantly, this flow satisfies the Holographic Wess–Zumino consistency condition, and in fact, the structure of the flow equation is protected by this condition. In other words, the flow equations receiving no corrections involving gradients or other operators formed from higher powers of the stress tensor is a consequence of the consistency conditions. To set this all up, it will help to set some conventions for two-dimensional conformal field theories on general backgrounds and their deformations.

3.1.1 2D conformal field theories on curved backgrounds

The conformal field theories under consideration are those living in two-dimensional spacetime. For our purposes, we will work with Euclidean conformal field theories. We will denote the action of the conformal field theory as S_{CFT} and the partition function reads

$$Z_{CFT}[g] = \int \mathcal{D}\Phi e^{-S_{CFT}[\Phi;g]}. \tag{3.1}$$

Here $g_{\mu\nu}$ denotes the metric tensor on the two-dimensional space on which the conformal field theory lives. The Φs refer to fundamental fields of arbitrary spin in principle. They needn't be specific for the rest of the discussion.

Putting the theory on an arbitrary curved background with metric $g_{\mu\nu}$ means that conformal symmetry will be anomalous. The anomaly itself is the sole source of scale dependence in this theory, and hence it will be useful to isolate said scale dependence. It will be useful to note that the metric can be decomposed as follows:

$$g_{\mu\nu} = e^{2\varphi(x)}\hat{g}_{\mu\nu},$$

where $\varphi(x)$ is referred to as the Liouville field. The partition function reflects this decomposition in the following manner:

$$Z_{CFT}[g = e^{2\varphi}\hat{g}] = e^{S_P[\varphi;\hat{g}]} Z^*_{CFT}[\hat{g}]. \tag{3.2}$$

The scale dependence rests in the Polyakov action

$$S_P[\varphi;\hat{g}] = \int \mathrm{d}^2x\sqrt{\hat{g}}\left(\varphi\Box\varphi - \varphi\bar{R}(\hat{g})\right).$$

The Ricci scalar R for g and that of $\hat{g}$ (denoted $\bar{R}(\hat{g})$) are related in the following manner:

$$R(g = \hat{g}e^{2\varphi(x)}) = e^{2\varphi}(\bar{R}(\hat{g}) + 2\Box\varphi)$$

The defining property of the conformal field theory is the Ward identity corresponding to Weyl invariance which reads:

$$\left(\frac{\delta}{\delta\varphi} - \frac{c}{24\pi}e^{-2\varphi}R(g)\right) Z_{CFT}[g] = 0. \tag{3.3}$$

Here, c denotes the central charge.

The purpose of isolating the φ dependence is to highlight the fact that the Polyakov action is solely responsible for the scale anomaly:

$$\frac{\delta S_p[\varphi;\hat{g}]}{\delta\varphi} = -\frac{c}{24\pi}e^{-2\varphi}R(g), \tag{3.4}$$

where as $Z^*_{CFT}[\hat{g}]$ remains completely Weyl invariant.

Despite the Weyl anomaly, the aforementioned ward identity will still be referred to as the conformal Ward identity. The breaking of conformal symmetry which results from RG flows being triggered will alter the form of this Ward identity, and the form the identity then takes will define an exact RG flow equation.

3.1.2 The deforming operator $T\bar{T}$

The form of the deforming operator $\mathcal{O}$ (which was referred to previously as $T\bar{T}$, and shall continue to use these two notations interchangeably) on flat space with complex coordinates $(z,\bar{z})$ is

$$\mathcal{O} = T\bar{T} - \frac{1}{4}\Theta^2,$$

where $T, \bar{T}, \Theta$ stand for the holomorphic, antiholomorphic components and trace of the energy-momentum tensor of the theory. In the limit where the theory is indeed conformal, $\mathcal{O} = T\bar{T}$. Zamalodchikov in [143],[157] proved that this deformation is integrable. This means that it is one of an infinite number of mutually commuting conserved charges. It was also shown that in the flat space limit for a wide class of slowly varying, translation invariant states, the following factorization property holds for the expectation value of this operator:

$$\langle\mathcal{O}\rangle = \langle T\rangle\langle\bar{T}\rangle - \langle\Theta\rangle^2. \tag{3.5}$$

This will be useful in constructing the bulk dual to this theory. Clearly, this operator is irrelevant, and as the above factorization property suggests, it is of dimension four. Thus the coupling of this operator in the action of the perturbed field theory is of mass dimension -2. It was shown in [51] that deforming a two dimensional quantum field theory by this operator on flat space is equivalent to coupling the theory to Jackiw–Teitelboim gravity. This is a very interesting observation but for the purposes of this chapter, i.e. to construct the off shell Einstein-Hilbert action in the holographically dual bulk theory, it will be necessary to put the theory we are interested in on an arbitrary curved background. In that case, $\mathcal{O}$ in a more covariant form is given by

$$\mathcal{O} = -\frac{1}{8}\left(T_{\mu\nu}T^{\mu\nu} - (T^{\alpha}_{\alpha})^2\right), \tag{3.6}$$

it can also be rewritten by noting that

$$T_{\mu\nu}T^{\mu\nu} - (T^{\alpha}_{\alpha})^2 = G^{\mathrm{dW}}_{\mu\nu\rho\sigma}T^{\mu\nu}T^{\rho\sigma},$$

where the de Witt super metric is defined as

$$G^{\mathrm{dW}}_{\mu\nu\rho\sigma} = (g_{\mu(\rho}g_{\sigma)\nu} - g_{\mu\nu}g_{\rho\sigma}). \tag{3.7}$$

The desired factorization property (3.5) will now take the form

$$\langle G^{\mathrm{dW}}_{\mu\nu\rho\sigma}T^{\mu\nu}T^{\rho\sigma}\rangle = G^{\mathrm{dW}}_{\mu\nu\rho\sigma}\langle T^{\mu\nu}\rangle\langle T^{\rho\sigma}\rangle.$$

And in order for the dual theory to indeed be general relativity in three dimensions, we will see that the above property has to be exact in addition to the fact that no operator other than $\mathcal{O}$ is generated to leading order in the coupling in front of it in conformal perturbation theory. In the next section, we will show that there is a natural consistency condition that the quantum field theory must satisfy in order for these hopes to be realized. The Energy Spectrum Here, we will briefly review the calculation of the energy spectrum of a $T\bar{T}$ deformed CFT on a cylinder. This is one of the first quantities that was computed in the literature ([**?**], [32]) in the deformed theory, and it sheds light on some very fascinating features of the deformation.

Given that the TT deformation preserves space and time translation invariance, a natural basis of states to consider in the theory is that of the Hamiltonian and momentum operators' simultaneous eigenstates. They are denoted as:

$$H|n\rangle = E_n|n\rangle,\ \ P|n\rangle = \frac{2\pi J_n}{L}|n\rangle. \tag{3.8}$$

Here L is the circumference of the spatial circle. In the deformed theory, the Hamiltonian satisfies the flow equation:

$$\partial_\mu\langle H\rangle = \int \mathrm{d}x\langle\mathcal{O}\rangle \tag{3.9}$$

which follows form a similar flow equation that the partition function of the theory on a general background satisfies. The expectation value on the right hand side is computed in the energy eigenbasis, and the left hand side too can be expanded in said basis:

$$\sum_n \partial_\mu E_n = \sum_n L\langle n|\mathcal{O}|n\rangle.$$

Therefore, we see that term by term we have

$$\partial_\mu E_n = \langle n|\mathcal{O}|n\rangle. \tag{3.10}$$

Thanks to the factorization result, which we know holds on translation invariance backgrounds, this expectation value can be rewritten as

$$\begin{aligned}\langle n|\mathcal{O}|n\rangle &= \left(\langle n|T|n\rangle\langle n|\bar{T}|n\rangle - (\langle n|\Theta|n\rangle)^2\right) \\ &= -\frac{1}{4}\left(\langle n|T_{tt}|n\rangle\langle n|T_{xx}|n\rangle - (\langle n|T_{tx}|n\rangle)^2\right).\end{aligned} \tag{3.11}$$

On the cylinder, we know what the individual components of the energy momentum tensor are in terms of energy, pressure and momentum:

$$\langle n|T_{tt}|n\rangle = \frac{E_n}{L},\ \langle n|T_{xx}|n\rangle = \partial_L E_n,\ \langle n|T_{xt}|n\rangle = \frac{2\pi i J_n}{L^2}. \tag{3.12}$$

The flow equation therefore becomes the following equation for the energy levels:

$$\partial_\mu E_n = -\frac{1}{4}\left(E_n\partial_L E_n + \frac{P_n^2}{L^2}\right). \tag{3.13}$$

This equation resembles the Burgers equation from hydrodynamics but without a viscosity like term. The solution to this equation, in the case where a CFT is being deformed is given by

$$E_n(\mu, L) = \frac{2L^2}{\mu}\left(1 - \sqrt{1 \mp \frac{2\pi\mu}{L^2}\left(\Delta_n + \bar{\Delta}_n - \frac{c}{12}\right) + \frac{\pi^2\mu^2}{L^4}(\Delta_n - \bar{\Delta}_n)^2}\right). \tag{3.14}$$

Here the undeformed energy is given by $E_o = \Delta_n + \bar{\Delta}_n - \frac{c}{24\pi}$ and the momentum eigenvalues are the same in the deformed and undeformed theories.

Note that there are two solutions, corresponding to the two signs to choose from. To match with calculations of quantities in AdS_3, it turns out that we have to choose the solution with a positive sign for μ. Indeed, this solution isn't real-valued for all values of n if we fix μ and it sees a square root singularity as a function of n. This level is to be interpreted as demarcating a cutoff, which is the interpretation that the authors of [114] proposed.

Moving beyond this paradigmatic example, and back to the task of applying QRG to a large c CFT on general backgrounds, it will help to first introduce the Callan–Symanzik equation for the deformed theory. This is needed since the definition involving the flow of the Hamiltonian works only when there is a time-like killing vector on the background.

3.1.3 The local renormalization group equation

The theory of interest in the discussion to follow has the following action

$$S[\Phi; g, \mu] = S_{CFT}[\Phi; g] + \frac{\mu}{16} \int \mathrm{d}^2 x \sqrt{g} G^{\mathrm{dW}}_{\mu\nu\alpha\beta} T^{\mu\nu} T^{\alpha\beta}, \tag{3.15}$$

where μ is kept infinitesimal for the rest of the discussion. This theory is no longer conformal, but depends on an additional scale through μ. The geometerization of the renormalization group flow this operator generates will entail said flow being mapped to Einstein's equation in a three dimensional (Euclidean) spacetime.

The aim here is to write down a Callan–Symanzik like equation describing the response of the generating functional to local Weyl transformations. These Weyl transformations are to be seen as a continuum generalization of the blocking transformations of Kadanoff, so they encode coarse-graining but in a space-dependent manner.

To begin, the partition function of the deformed field theory takes the form

$$Z_{QFT}[g, \mu] = Z_{CFT}[g] \langle e^{-\frac{\mu}{2} \int \mathrm{d}^2 x \sqrt{g} G^{\mathrm{dW}}_{\mu\nu\rho\sigma} T^{\mu\nu} T^{\rho\sigma}} \rangle_{CFT}. \tag{3.16}$$

The key results of this subsection shall all follow from the above expression where μ is considered to be small. It will thus be very important to state the assumptions that go behind this expression.

The most important one is that the only scale in the quantum field theory is μ. This implicitly assumes that no operators of higher dimensions suppressed by other scales enter into the RG flow of the quantum field theory. We assume that the theory possesses a large central charge c. Given that the energy-momentum tensor and its derivatives, i.e. the descendants of the identity operator close under the operator product expansion with each other, the higher dimension operators possibly entering with scales other than μ

are all composite operators of the energy-momentum tensor. In the next section, we will describe what the necessary condition is in order to protect the theory of interest from such corrections while retaining only $\mathcal{O}$. For the time being, we will continue with more the more stringent assumption that μ is in fact the only scale available in the quantum field theory, and also that $g_{\mu\nu}(x)$ to vary slowly with x.

We will start by introducing the Liouville field:

$$Z_{QFT}[g,\mu] = Z^*_{CFT}[\hat{g}]e^{S_P[\hat{g},\varphi]}\langle e^{-\frac{\mu}{2}\int \mathrm{d}^2x\sqrt{\hat{g}}e^{-2\varphi}G^{\mathrm{dW}}_{\mu\nu\rho\sigma}T^{\mu\nu}T^{\rho\sigma}}\rangle_{CFT}.$$

The modified scale Ward identity that takes into account the breaking of conformal invariance by the deformation-

$$-\frac{e^{-2\varphi}}{4}\frac{\delta\ln Z_{QFT}[g,\mu]}{\delta\varphi} = \langle T^\alpha_\alpha\rangle = -\frac{c}{96\pi}R(\varphi,\hat{g}) - \frac{\mu}{2}\langle G^{\mathrm{dW}}_{\mu\nu\rho\sigma}(\varphi,\hat{g})T^{\mu\nu}T^{\rho\sigma}\rangle,$$

which shows how in addition to the anomaly, how the operator $\mathcal{O}$ drive the RG flow, at least to the lowest order in μ. Another use for the Liouville field is to act as a compensator for Weyl transformations- where transformations of the form $g_{\alpha\beta}\to e^{2\sigma}g_{\alpha\beta}$, this can be compensated through $\varphi\to\varphi-\sigma$. Thus the above equation can be written as

$$\delta_\sigma\ln Z = \int \mathrm{d}^2x\sqrt{g}\sigma\langle T^\alpha_\alpha\rangle = \int \mathrm{d}^2x\sqrt{g}\sigma\left(-\frac{c}{96\pi}R(g) - \frac{\mu}{2}\langle G^{\mathrm{dW}}_{\mu\nu\rho\sigma}T^{\mu\nu}T^{\rho\sigma}\rangle\right), \tag{3.17}$$

where $\delta_\sigma := \int \mathrm{d}^2x\sqrt{g}\sigma g_{\mu\nu}\frac{\delta}{\delta g_{\mu\nu}}$. Notice that despite the split between the Liouville field and the metric $\hat{g}_{\mu\nu}$ the right-hand side of the above equation only sees φ and $\hat{g}_{\mu\nu}$ in their combination $g_{\mu\nu}$. This is an equation which describes the response of the generating functional to the change of local scale encoded in the Weyl transformation $g_{\mu\nu}\to e^{-2\sigma(x)}g_{\mu\nu}$. This Weyl transformation can be seen as a generalization of the blocking transformation Kadanoff introduced on the lattice. In principle, if other sources were considered, then functional derivatives with respect to such sources will result in correlation functions of the operators to which they couple. The response of such correlation functions under the aforementioned coarse-graining transformations corresponds to inserting the trace of the energy-momentum tensor into said correlates which one can deduce from the above equation. Thus this equation is the local Callan–Symanzik equation.

Now, in the large c limit and taking $\hat{g}_{\mu\nu}=\eta_{\mu\nu}$, the factorisation property of Zamalodchikov can be utilised:

$$\langle\Theta\rangle = -\frac{c}{96\pi}R(\varphi) - \frac{\mu}{2}\left(\eta_{\mu(\rho}\eta_{\sigma)\nu} - \eta_{\mu\nu}\eta_{\rho\sigma}\right)\langle T^{\mu\nu}\rangle\langle T^{\rho\sigma}\rangle. \tag{3.18}$$

Notice that $\frac{1}{2}\left(\eta_{\mu(\rho}\eta_{\sigma)\nu} - \eta_{\mu\nu}\eta_{\rho\sigma}\right)\langle T^{\mu\nu}\rangle\langle T^{\rho\sigma}\rangle = \langle T\rangle\langle\bar{T}\rangle - \langle\Theta\rangle^2$ in the $(z, \bar{z})$ co-ordinates.

The local RG equation (3.18) can be seen as the defining property that the quantum field theory perturbed by $\mathcal{O}$ needs to satisfy. But we must remember that several assumptions were made to derive this equation, and the aim of the next section will be to present a consistency condition which will allow one to relax a few of these assumptions and also to allow the construction of the Einstein-Hilbert action of the holographically dual bulk theory. In other words, this consistency condition too must be included in the very definition of the theory we are interested in.

The last subsection to follow will describe why (3.18) would ensure that it is indeed general relativity operating in the bulk theory.

3.1.4 What to aim at: The holographic interpretation of the local Callan-Symanzik equation

If the field theory under consideration were to possess a holographic dual, then the generating function can be equated, at large c to the on shell classical Einstein–Hilbert action for gravity in $2+1$ dimensions (denoted S_c),

$$\ln Z_{QFT}[g;\mu] = -\frac{c}{24\pi}S_c[g,\mu] \tag{3.19}$$

The local Callan–Symanzik equation for such a theory takes the form

$$\frac{\delta S_c}{\delta\varphi} = e^{-2\varphi}R(\varphi,\hat{g}) - \frac{c\mu}{24\pi}\left(\frac{\delta S_c}{\delta\hat{g}_{\mu\nu}}\frac{\delta S_c}{\delta\hat{g}_{\rho\sigma}} - \frac{1}{2}\left(\frac{\delta S_c}{\delta\varphi}\right)^2\right). \tag{3.20}$$

This is nothing but the Einstein-Hamilton-Jacobi equation for pure (Euclidean) general relativity in $2+1$ dimensions provided one sets $\mu = \frac{24\pi}{c}$. This observation was made in [114], where the above expression first appeared. Notice however that in contrast to (3.18), the above equation holds away from $\hat{g}_{\mu\nu} = \eta_{\mu\nu}$, and beyond the leading order in the gradient expansion. It is instructive to rewrite the above equation in terms of the metric $g_{\mu\nu}$:

$$g_{\mu\nu}\frac{\delta S_c}{\delta g_{\mu\nu}} = -R(g) - G^{\mathrm{dW}}_{\mu\nu\rho\sigma}\frac{\delta S_c}{\delta g_{\mu\nu}}\frac{\delta S_c}{\delta g_{\rho\sigma}}. \tag{3.21}$$

Recalling that $\frac{\delta S_c}{\delta g_{\mu\nu}} = \langle T^{\mu\nu}\rangle$, and comparing the above form of the local Callan–Symanzik equation with the general form (3.17), we see that the factorisation condition

$$\langle G^{\mathrm{dW}}_{\mu\nu\rho\sigma}T^{\mu\nu}T^{\rho\sigma}\rangle = G^{\mathrm{dW}}_{\mu\nu\rho\sigma}\langle T^{\mu\nu}\rangle\langle T^{\rho\sigma}\rangle$$

holds for these theories that possess holographic duals. This is what we meant earlier when it was mentioned that such a factorisation property will be crucial importance.

The remainder of this chapter is dedicated to deriving the above result (3.19) starting from the $T\bar{T}$ deformed boundary CFT by reorganizing the RG flow. Such efforts to construct the bulk theory from the RG flow of the boundary theory go under the heading of the 'quantum renormalization group' (QRG) [103], [102]. An intended by-product of such an effort will be to clearly understand what criteria the theories which possess holographic bulk duals should possess.

3.2 Constructing the Bulk theory

In this section, we wish to start only from the two-dimensional field theory deformed by the operator $\mathcal{O}$ and construct from that the off-shell, Einstein Hilbert action in three dimensions. We'll show that the criterion for such a construction to work will hinge on a very particular composition property of the coarse-graining transformations.

In other words, we wish to show what criterion needs be met in order for the $\mathcal{O}$ deformed quantum field theory to remain protected under corrections involving operators of higher dimensions including gradients, etc. at least in the large c limit and in conformal perturbation theory. In other words, many of the assumptions from the previous section can be dropped provided that this criterion is met.

The idea behind the holographic duality is that the evolving bulk fields (are a subset of) the sources of the boundary operators somehow granted dynamics. More specifically, the radial evolution of the bulk fields is equated with the renormalization group flow. In the case of interest here, the conformal invariance is broken by a composite operator involving solely the energy-momentum tensor, whose source is the metric. On the other side of the duality, the only evolving field will thus be the metric, and so one can expect pure gravity to be the theory at play in the bulk. In the first subsection, we describe how to grant dynamics to sources of composite operators in the theory. In the case of interest, we have just the metric and that couples to the energy-momentum tensor.

3.2.1 The dual theory

The end result of the QRG procedure applied to the theory we are interested in, which is a two dimensional conformal field theory with large central charge c deformed by the $T\bar{T}$

operator. The path integral manifestation of the anomalous Ward identity is that it allows us to answer the question of how the functional integral responds to the infinitesimal Weyl transformations $g_{\mu\nu} \to e^{\sigma^{(0)}(x)\delta z} g_{\mu\nu}$, where δz is an infinitesimal constant. This is given by

$$Z[g_{\mu\nu}] = \int \mathcal{D}\Phi e^{-S_{QFT}[\Phi] - \delta z \int_x \sigma^{(0)}(x)(T^{\alpha}_{\alpha} - \frac{\mu}{16} G^{\rm dW}_{\mu\nu\rho\eta} T^{\mu\nu} T^{\rho\eta} - \frac{c}{24\pi} R + \cdots)}, \tag{3.22}$$

and the $(\cdots)$ stands for the potential corrections coming from potential higher dimension operators that might arise. This was the possibility alluded to in the previous section. In order to show how the consistency conditions we will later impose protect the form of the local Callan–Symanzik equation that can be interpreted as the Einstein Hamilton Jacobi equations, we will begin by assuming that higher order corrections too can in principle appear but all but the lowest order term corresponding to $\mathcal{O}$ will not satisfy the aforementioned criterion. To begin however, let $f(T^{\mu\nu}, g_{\mu\nu})$ encode all these corrections i.e.

$$f(T^{\mu\nu}, g_{\mu\nu}) := \frac{\mu}{16} G^{\rm dW}_{\mu\nu\rho\eta} T^{\mu\nu} T^{\rho\eta} - \frac{c}{24\pi} R + \cdots$$

Now in order to grant the metric dynamics, we use the following trick:

$$\int \mathcal{D}\Phi \mathcal{D}g^{(0)}_{\mu\nu} \delta(g^{(0)}_{\mu\nu} - g_{\mu\nu}) e^{-S_{QFT}[\Phi; g] - \delta z \int_x \sigma^{(0)}(x)(T^{\alpha}_{\alpha} - f(T^{\mu\nu},\ g^{(0)}_{\mu\nu}))}.$$

It will also be useful to write think of $T^{\mu\nu}$ as a normalised functional derivative with respect to the metric acting on the generating functional:

$$\int \mathcal{D}\Phi \mathcal{D}g^{(0)}_{\mu\nu} \delta(g^{(0)}_{\mu\nu} - g_{\mu\nu}) \exp\left\{-\delta z \int_x \sigma^{(0)}(x) \left(\frac{24\pi}{c} g^{(0)}_{\mu\nu} \frac{\delta}{\delta g^{(0)}_{\mu\nu}} - f\left(\frac{24\pi}{c} \frac{\delta}{\delta g^{(0)}_{\mu\nu}}, g^{(0)}_{\mu\nu}\right)\right)\right\} e^{-S_{QFT}[\Phi; g^{(0)}]}.$$

and then we exponentiate the delta function

$$\int \mathcal{D}\Phi \mathcal{D}g^{(0)}_{\mu\nu} \mathcal{D}\pi^{(0)\mu\nu} e^{-\frac{c}{24\pi}\pi^{(0)\mu\nu}(g^{(0)}_{\mu\nu} - g_{\mu\nu})} \times$$

$$\exp\left\{-\delta z \int_x \sigma^{(0)}(x) \left(\frac{24\pi}{c} g^{(0)}_{\mu\nu} \frac{\delta}{\delta g^{(0)}_{\mu\nu}} - f\left(\frac{24\pi}{c} \frac{\delta}{\delta g^{(0)}_{\mu\nu}}, g^{(0)}_{\mu\nu}\right)\right)\right\} e^{-S_{QFT}[\Phi; g^{(0)}]}.$$

At this stage, we can also make note of the fact that the theory coupled to an arbitrary background geometry $g^{(0)}_{\mu\nu}$ should be invariant under diffeomorphism transformations $g^{(0)}_{\mu\nu} \to g^{(0)}_{\mu\nu} + \delta z(\nabla_{(\mu}\xi^{(0)}_{\nu)})$. This invariance manifests itself through the diffeomorphism Ward identity which in the path integral appears as:

$$\int \mathcal{D}\Phi \mathcal{D}g^{(0)}_{\mu\nu} \mathcal{D}\pi^{(0)\mu\nu} e^{-\frac{c}{24\pi}\pi^{(0)\mu\nu}(g^{(0)}_{\mu\nu} - g_{\mu\nu})} \times$$

$$\exp\left\{-\delta z \int_x \sigma^{(0)}(x)\left(\frac{24\pi}{c} g^{(0)}_{\mu\nu}\frac{\delta}{\delta g^{(0)}_{\mu\nu}} - f\left(\frac{24\pi}{c}\frac{\delta}{\delta g^{(0)}_{\mu\nu}}, g^{(0)}_{\mu\nu}\right)\right) + \frac{24\pi}{c}(\nabla_{(\mu}\xi^{(0)}_{\nu)})\frac{\delta}{\delta g^{(0)}_{\mu\nu}}\right\} e^{-S_{QFT}[\Phi;g^{(0)}]}.$$

We can then functionally integrate by parts so that the functional derivatives with respect to the metric $g^{(0)}_{\mu\nu}$ get replaced by $\pi^{(0)\mu\nu}$, and we find

$$\int \mathcal{D}\Phi \mathcal{D}g^{(0)}_{\mu\nu}\mathcal{D}\pi^{(0)\mu\nu} e^{-\frac{c}{24\pi}\pi^{(0)\mu\nu}(g^{(0)}_{\mu\nu}-g_{\mu\nu})}\times$$

$$\exp\left\{-\delta z\left(\frac{c}{24\pi}\right)\int_x \sigma^{(0)}(x)\left(g^{(0)}_{\mu\nu}\pi^{(0)\mu\nu} - f\left(\pi^{(0)\mu\nu}, g^{(0)}_{\mu\nu}\right)\right) + (\nabla_{(\mu}\xi^{(0)}_{\nu)})\pi^{(0)\mu\nu}\right\} e^{-S_{QFT}[\Phi;g^{(0)}]}.$$

We see that the partition function of the original quantum field theory can be recollected although the theory now lives on the geometry described by metric $g^{(0)}_{\mu\nu}$:

$$\int \mathcal{D}g^{(0)}_{\mu\nu}\mathcal{D}\pi^{(0)\mu\nu} e^{-\frac{c}{24\pi}\pi^{(0)\mu\nu}(g^{(0)}_{\mu\nu}-g_{\mu\nu})}\times$$

$$\exp\left\{-\delta z\left(\frac{c}{24\pi}\right)\int_x \sigma^{(0)}(x)\left(g^{(0)}_{\mu\nu}\pi^{(0)\mu\nu} - f\left(\pi^{(0)\mu\nu}, g^{(0)}_{\mu\nu}\right)\right) + (\nabla_{(\mu}\xi^{(0)}_{\nu)})\pi^{(0)\mu\nu}\right\} Z_{QFT}[g^{(0)}].$$

Now this process of performing infinitesimal Weyl transformations can be iterated say k times until we find

$$\int \prod_{i=0}^{k} \mathcal{D}g^{(i)}_{\mu\nu}\mathcal{D}\pi^{(i)\mu\nu} e^{-\frac{c}{24\pi}k\delta z\sum_{i=0}^{k}\pi^{(i)\mu\nu}\frac{(g^{(i)}_{\mu\nu}-g^{(i-1)}_{\mu\nu})}{\delta z}}\times$$

$$\exp\sum_{i=0}^{k}\left\{-\delta z\left(\frac{c}{24\pi}\right)\int_x \sigma^{(i)}(x)\left(g^{(i)}_{\mu\nu}\pi^{(i)\mu\nu} - f\left(\pi^{(i)\mu\nu}, g^{(i)}_{\mu\nu}\right)\right) + (\nabla_{(\mu}\xi^{(i)}_{\nu)})\pi^{(i)\mu\nu}\right\} Z_{QFT}[g^{(k)}].$$

The continuum limit can now be taken where $\delta z \to 0$, $k\delta z\sum_{i=0}^{k} := \int^{z_*} \mathrm{d}z$, and the collections of fields $(g^{(i)}_{\mu\nu}(x), \pi^{(i)\mu\nu}(x), \sigma^{(i)}(x), \xi^{(i)\mu}(x))$ now are replaced by fields with z dependence $(g_{\mu\nu}(x,z), \pi^{\mu\nu}(x,z), \sigma(x,z), \xi^{\mu}(x,z))$. Then we find the emergence of a 'bulk theory' with action S_B:

$$\int \mathcal{D}g_{\mu\nu}(x,z)\mathcal{D}\pi^{\mu\nu}(x,z)\mathcal{D}\sigma(x,z)\mathcal{D}\xi_{\mu}(x,z) e^{-\frac{c}{24\pi}S_B[g_{\mu\nu}(x,z),\pi^{\mu\nu}(x,z),\alpha(x,z),\xi^{\mu}(x,z)]} Z_{QFT}[g(x,z=z_*)].$$

Here, the $\sigma(x,z)$ and $\xi(x,z)$ are also integrated over because they appear only linearly in the action and are thus Lagrange multipliers.

The form of the bulk action is given by

$$S_B = \int \mathrm{d}^2x\mathrm{d}z\sqrt{g}\left(\pi^{\mu\nu}(x,z)\dot{g}_{\mu\nu}(x,z) - H_B(\pi^{\mu\nu}, g_{\mu\nu}, \sigma, \xi^\mu)\right). \tag{3.23}$$

We see that the normalisation of the functional derivatives that get traded for the momenta and thus the overall normalisation of the action is chosen such that the large c limit of this theory does indeed give the semiclassical limit where the partition function can be evaluated in the saddle point approximation. Now, the bulk Hamiltonian reads

$$H_B(\pi^{\mu\nu}, g_{\mu\nu}, \sigma, \xi^\mu) = \sigma(x,z)H(\pi^{\mu\nu}, g_{\mu\nu}) + \xi^\mu(x,z)H_\mu(\pi^{\mu\nu}, g_{\mu\nu}),$$

which is a sum of just constraints. These constraints are the dual versions of the anomalous Ward identity for broken Weyl invariance and for diffeomorphism invariance on the side of the quantum field theory.

$$H(\pi^{\mu\nu}, g_{\mu\nu}) = \mathrm{tr}\pi - f(\pi^{\mu\nu}, g_{\mu\nu}), \tag{3.24}$$

$$H_\mu(\pi^{\mu\nu}, g_{\mu\nu}) = -2\nabla_\rho \pi^\rho_\mu. \tag{3.25}$$

The question then is what form the function $f(\pi^{\mu\nu}, g_{\mu\nu})$ can take, and what dictates it's structure. In the following subsection, we will show how the demand that the Poisson algebra of these constraints agrees with that of general relativity fixes completely the form of this function and when translated back to the field theory, this will correspond exactly to the desirable form of the anomalous Ward Identity.

3.2.2 Leveraging the consistency conditions

Note that the Holographic Wess–Zumino (WZ) consistency conditions:

$$0 = [\Delta_\sigma, \Delta_{\sigma'}]\ln Z_{QFT}[g]|_{c\to\infty} = \int \mathrm{d}^2x\sqrt{g}g^{\mu\nu}(\sigma\partial_\nu\sigma' - \sigma'\partial_\nu\sigma)\langle\nabla_\kappa T^\kappa_\nu\rangle. \tag{3.26}$$

applied to the present context implies that we must have:

$$f(\pi^{\mu\nu}, g_{\mu\nu}) = \frac{1}{\sqrt{g}}G^{\mathrm{dW}}_{\mu\nu\rho\sigma}\pi^{\mu\nu}\pi^{\rho\sigma} - \sqrt{g}R. \tag{3.27}$$

As a reminder, the reason for this is that demanding the Poisson bracket algebra of the smeared constraints takes the form

$$\{H_\mu(\xi^\mu), H_\mu(\chi^\mu)\} = H_\mu([\xi,\chi]^\mu) \tag{3.28}$$
$$\{H(\sigma), H_\mu(\xi^\mu)\} = H(\xi^\mu \nabla_\mu \sigma) \tag{3.29}$$
$$\{H(\sigma), H(\sigma')\} = H_\mu(g^{\mu\nu}(\sigma\partial_\nu\sigma' - \sigma'\partial_\nu\sigma)), \tag{3.30}$$

will fix the form of the total Hamiltonian, and consequently of both H and H_μ to be of the form

$$H = \frac{1}{\sqrt{g}} G^{\rm dW}_{\mu\nu\rho\sigma}\pi^{\mu\nu}\pi^{\rho\sigma} - \sqrt{g}R + {\rm tr}\pi \tag{3.31}$$
$$H_\mu = -2\nabla_\nu \pi^\nu_\mu. \tag{3.32}$$

When these conditions are satisfied, they ensure that the form of the exact RG equation that corresponds to the Hamilton–Jacobi equation in the bulk gravitational theory.

3.2.3 The Hamilton–Jacobi Equation

Given the normalization of the bulk action that has been chosen, we see that the role of $\hbar$ in the bulk theory is played by $\frac{1}{c}$. In the large c limit, this is small and the functional integral can be evaluated in the saddle point approximation. In other words, the bulk theory is in its semiclassical limit. This implies that the relationship

$$\lim_{c\to\infty} \ln Z_{QFT}[g] \propto \frac{c}{24\pi} S_B^{o.s.}, \tag{3.33}$$

holds, where $o.s.$ superscript denotes that the bulk action is taken on- shell. The Hamiltonian constraint then takes the form of the Einstein-Hamilton-Jacobi equation

$$g_{\mu\nu}\frac{\delta S_B^{o.s}}{\delta g_{\mu\nu}} = -R(g) - G^{\rm dW}_{\mu\nu\rho\sigma}\frac{\delta S_B^{o.s}}{\delta g_{\mu\nu}}\frac{\delta S_B^{o.s}}{\delta g_{\rho\sigma}}.$$

Which is simply the Hamiltonian constraint re-written with the identification $\pi^{\mu\nu} = \frac{\delta S_B^{o.s}}{\delta g_{\mu\nu}}$ being made. Given the identification (3.33), this implies that:

$$\pi^{\mu\nu} = \lim_{c\to\infty}\frac{24\pi}{c}\frac{\delta}{\delta g_{\mu\nu}}\ln Z_{QFT}[g] = \langle T^{\mu\nu}\rangle.$$

As promised, this is nothing but the factorised exact RG equation

$$\langle T^\alpha_\alpha\rangle = -R(g) - G^{\rm dW}_{\mu\nu\rho\sigma}\langle T^{\mu\nu}\rangle\langle T^{\rho\sigma}\rangle, \tag{3.34}$$

with no corrections.

In the following subsection, we will describe where the cosmological constant lies in this construction.

3.2.4 The cosmological constant

In order to bring the Hamiltonian constraint to a more standard form, it is necessary to eliminate the term linear in the momentum. This will require performing a canonical transformation:

$$\pi^{\mu\nu} \to \pi^{\mu\nu} - \frac{\delta C[g]}{\delta g_{\mu\nu}}, \tag{3.35}$$

with generating functional $C[g] = \frac{24\pi}{c} \int \mathrm{d}^2 x \sqrt{g}$ proportional to the volume. The price of performing this canonical transformation is that the momentum independent part of the scalar constraint is modified as follows:

$$\sqrt{g}R(g) \to \sqrt{g}\,(R(g)) - \frac{1}{2\sqrt{g}} G^{\text{d.W}}_{\mu\nu\alpha\beta} \frac{\delta C[g]}{\delta g_{\mu\nu}} \frac{\delta C[g]}{\delta g_{\alpha\beta}} = \sqrt{g}(R(g) - 2). \tag{3.36}$$

Thus we see that the net effect of the canonical transformation is to add a cosmological constant to the theory (in units where it has been set to 1). The overall normalisation of the action is thus consistent with the identification $c \propto \frac{l}{G}$.

Also note that the momenta $\pi^{\mu\nu}$ can now be integrated out in the bulk partition function, which in the saddle point approximation is nothing but the Legendre transform. We then find that

$$S_B = \frac{c}{24\pi} \int \mathrm{d}^3 x \; ({}^{(3)}R(\gamma) - 2)\sqrt{\gamma}. \tag{3.37}$$

which is nothing but the Einstein–Hilbert action for the three dimensional metric γ_{ab} on the bulk space with negative cosmological constant set to 1.

3.3 Epilogue

In this section, we showed the HWZ conditions can be used as a criterion to select deformed CFTS which can be mapped to the dual of a bulk theory cutoff at a finite radial slice. This is provided we restrict our attention to the identity module of two-dimensional CFTs picks out the $T\bar{T}$ deformation at large c.

However, the form of the $T\bar{T}$ deformed flow equation mirroring that of the Hamiltonian constraint at large c is but a limiting case of how this flow equation even at finite c is an exact re-writing of the Wheeler de Witt equation, which is a quantization of the Hamiltonian constraint. This mapping will be described in one of the sections to follow. It somehow turns out that the $T\bar{T}$ flow satisfies the HWZ conditions even away from the limit where the bulk theory is described by classical gravity.

3.4 Higher dimensional Examples

In this section, we will describe higher-dimensional generalizations of the $T\bar{T}$ deformation that play the role of the dual theories to gravity (and coupled matter) in AdS space, cutoff at some finite radial slice. In particular, we will look once again at AdS_5. Furthermore, we will connect these deformations to earlier results of holographic RG, and to the so-called generalized gradient flow.

The first attempt at such a generalization was done in [80], where the authors define the effective field theory's generating functional through the following flow equation (in two, three or four dimensions):

$$\frac{\partial \ln Z_{eff}}{\partial \lambda} = -\int \mathrm{d}^D x \sqrt{g} X, \tag{3.38}$$

$$X = \left(T_{\mu\nu} + a_D r_c^{D-2} \tilde{C}_{\mu\nu}\right)^2 - \frac{1}{D-1}\left(T^\mu_\mu + a_d r_c^{D-2} \tilde{C}^\nu_\nu\right)^2 - \frac{r_c^D}{D\lambda}\left(\tilde{t}^\mu_\mu - \frac{\tilde{R}}{16\pi G} - a_D \tilde{C}^\mu_\mu\right). \tag{3.39}$$

Here, $T_{\mu\nu}$ denotes the energy momentum tensor, the quantity $\tilde{C}_{\mu\nu}$ depends on the curvature tensors of the background geometry with metric $g_{\mu\nu}$. The constant r_c denotes the radius in the bulk corresponding to the cutoff surface, and the constant a_D stands for $\frac{1}{8\pi G(D-2)}$. The re-scaled Ricci scalar is denoted $\tilde{R} = r_c^{D-2} R$ and the tensor $\tilde{t}_{\mu\nu}$ is a function of sources other than the metric that are also turned on.

A brief explanation for why the scale associated with these double trace deformations can also be seen as a cutoff in energy for the deformed holographic quantum field theory is necessary at this point. Following the authors of [80], consider the theory defined on a simple geometry, namely a square torus of length L, and let us momentarily turn off sources other than the metric.[1] Also assume that there is no momentum carried in the state in which the energy-momentum tensor's expectation values are evaluated. In this situation, the energy density is given by the expectation value of one of the components of the energy-momentum tensor. This can be integrated up to find the energy $E(L, \lambda)$ of the system and it is found to be

$$E(L, \lambda) = \frac{(D-1)L^{D-1}}{2D\lambda}\left(1 - \sqrt{1 - \frac{4D\lambda E_o(L)}{(D-1)L^{D-1}}}\right), \tag{3.40}$$

[1] This can be done more generally, as the authors of [80] show, however, for the purpose of illustrating the role of λ as a cutoff scale, this setting is sufficient

where $E_o(L)$ denotes the energy of the undeformed CFT. This function hits a square root singularity at $E^* = \frac{(D-1)L^{D-1}}{4D\lambda}$, after which the energies take imaginary values. The idea is to discard all energies above E^* and treat this value of the energy as a cutoff. In fact, this equation could also be used to write λ as $\frac{(D-1)L^{D-1}}{4DE^*}$ which clarifies its role as a cutoff scale.

This expression arises from taking the radial bulk Hamiltonian constraint in AdS space and identifying the metric on a constant radius hypersurface in the bulk with $r_c^2 g_{\mu\nu}$, the momentum conjugate to this metric with $\sqrt{g}((r_c^{2-D}T^{\mu\nu} - (D-1)g^{\mu\nu}(x, r=0)) + a_d\tilde{C}^{\mu\nu})$, and the parameter λ identified as

$$\lambda = \frac{4\pi G}{r_c D}. \tag{3.41}$$

This allows us to rewrite the derivative $\frac{\partial}{\partial\lambda}$ in terms of the derivative $\frac{\partial}{\partial r_c}$ in the left hand side of (3.38). For the full list of identifications, see [80]. The above equation is guaranteed to give rise to the right kind of bulk physics although it is unclear what the underlying coarse graining mechanism is that gives rise to such a flow.

However, we will present an alternative prescription to the one above, through which flow equations similar to the one above are derived for four dimensional, large N, holographic conformal field theories. The advantage of our approach is that the functions of background fields appearing in (3.38) are shown to arise from certain cancellations rather than being posited in the definition of the effective field theory. The route taken is to start from the definition of a conformal field theory deformed by certain double trace operators:

$$\frac{\partial \ln Z}{\partial\lambda} = \int \mathrm{d}^4x\sqrt{g}\left(-\mu\langle TT\rangle - \frac{\kappa}{2}\langle\mathcal{O}\mathcal{O}\rangle\right) - \mathcal{A}[g,\phi], \tag{3.42}$$

where $g_{\mu\nu}(x)$ and $\phi(x)$ are sources for the operator $T^{\mu\nu}$ and $\mathcal{O}(x)$ respectively. The functional $\mathcal{A}[g,\phi]$ is the integrated conformal anomaly. The reason why it appears here is that when the deformation parameters μ and κ are set to zero, the flow equation reduces to the statement that the conformal field theory in the presence of arbitrary background fields is anomalous under scale transformations. The scalar double trace deformation is formed from the single trace scalar operator $\mathcal{O}(x)$ which couples to source $\phi(x)$, and its expectation value is denoted by $\langle\mathcal{O}\mathcal{O}\rangle$. Similarly, $\langle TT\rangle$ denotes the expectation value of the following operator:

$$TT(x) \equiv \left(G_{\mu\nu\rho\sigma}T^{\mu\nu}T^{\rho\sigma}\right)(x), \tag{3.43}$$

where $G_{\mu\nu\rho\sigma} = g_{\mu(\rho}g_{\sigma)\nu} - \frac{1}{3}g_{\mu\nu}g_{\rho\sigma}$ is the de Witt supermetric in four dimensions.[2]

[2] We assume tentatively that an appropriate regularization prescription has been chosen to define these double trace operators but this will not feature in anything to follow because we will be using large N factorization to make sense of the expectation values of the operators in the above flow equation.

The operator (3.43) is the same as the one proposed by Taylor in [147] as the higher dimensional generalizations of the two dimensional $T\bar{T}$ deformation introduced in [143], [32].

In the large N limit, the expectation value of the double trace operators factorizes:

$$\langle TT\rangle = G_{\mu\nu\rho\sigma}\langle T^{\mu\nu}\rangle\langle T^{\rho\sigma}\rangle, \quad \langle \mathcal{O}\mathcal{O}\rangle = \langle\mathcal{O}\rangle^2, \tag{3.44}$$

and so the equation that we will use reads

$$\frac{\partial \ln Z[g,\phi]}{\partial\lambda} = -\int \mathrm{d}^4x\sqrt{g}\left(\mu G_{\mu\nu\rho\sigma}\langle T^{\mu\nu}\rangle\langle T^{\rho\sigma}\rangle + \frac{\kappa}{2}\langle\mathcal{O}\rangle^2\right) - \mathcal{A}[g,\phi]. \tag{3.45}$$

The left hand side can be rewritten as follows

$$\frac{\partial \ln Z[g,\phi]}{\partial\lambda} = \int \mathrm{d}^4x\sqrt{g}\left(\beta_{\mu\nu}(g,\phi)\frac{\delta}{\delta g_{\mu\nu}} + \beta_\phi(g,\phi)\frac{\delta}{\delta\phi}\right)\ln Z[g,\phi]. \tag{3.46}$$

where the flow functions are defined as:

$$\beta_{\mu\nu}(g,\phi) = \frac{\partial g_{\mu\nu}}{\partial\lambda}, \quad \beta_\phi(g,\phi) = \frac{\partial\phi}{\partial\lambda}. \tag{3.47}$$

Similar flow functions are defined in [128] although the beta functions there also depend on the expectation values of the deforming operators, and lead in a different manner to RG flow equations that can be mapped into the bulk Hamiltonian and momentum constraints.

Going back to our flow equation (3.46), the exercise now is to find an appropriate change of variables so that the equation

$$\int \mathrm{d}^4x\sqrt{g}\left(\beta_{\mu\nu}(g,\phi)\frac{\delta}{\delta g_{\mu\nu}} + \beta_\phi(g,\phi)\frac{\delta}{\delta\phi}\right)\ln Z[g,\phi] =$$

$$-\int \mathrm{d}^4x\sqrt{g}\left(\mu G_{\mu\nu\rho\sigma}\langle T^{\mu\nu}\rangle\langle T^{\rho\sigma}\rangle + \frac{\kappa}{2}\langle\mathcal{O}\rangle^2\right) - \mathcal{A}[g,\phi], \tag{3.48}$$

can be rewritten as the radial Hamiltonian of a gravitational theory in one higher dimension. This will require identifying the right change of variables, and furthermore, we will find that the flow functions (3.47) must take a very particular form.

3.4.1 Pure gravity in the bulk

Lets start by turning off the sources for the scalar operator. In this case, the flow equation (3.48) reads

$$\int \mathrm{d}^4x\sqrt{g}\beta_{\mu\nu}(g)\frac{\delta \ln Z[g]}{\delta g_{\mu\nu}} = \int \mathrm{d}^4x\sqrt{g}\sigma(x)\beta_{\mu\nu}(g)\langle T^{\mu\nu}\rangle =$$
$$= \int \mathrm{d}^4x\sqrt{g}\left(-\mu G_{\mu\nu\rho\sigma}\langle T^{\mu\nu}\rangle\langle T^{\rho\sigma}\rangle\right) - \mathcal{A}[g]. \tag{3.49}$$

The aim is to convert this into the radial Hamiltonian constraint of general relativity operating in one higher dimension. The idea is to identify the theory's background metric with the metric induced on a constant radius hypersurface and relate the one-point function of the energy-momentum tensor in a general background to the momentum conjugate to this metric.

The latter is what we will pay attention to here. The discrepency between the energy momentum tensor's one point function and the momentum is captured by the ambiguity in adding local counterterms. By counterterms here, we are referring to local functionals of the source, here the metric, and its derivatives that are added to the generating functional. These aren't added in order to cancel any divergences although in the limit where the regularization is removed, this would be their purpose. In other words:

$$\sqrt{g}\langle T^{\mu\nu}\rangle = \pi^{\mu\nu} + \frac{\delta S[g]}{\delta g_{\mu\nu}}. \tag{3.50}$$

Plugging this into the flow equation leads to the following equation:

$$\int \mathrm{d}^4x\sqrt{g}\left(\beta_{\mu\nu}\pi^{\mu\nu} + \beta_{\mu\nu}\frac{\delta S[g]}{\delta g_{\mu\nu}}\right) =$$
$$= \int \mathrm{d}^4x\sqrt{g}\left(-\frac{\mu}{\sqrt{g}}G_{\mu\nu\rho\sigma}\pi^{\mu\nu}\pi^{\rho\sigma} + 2\mu G_{\mu\nu\rho\sigma}\pi^{\mu\nu}\frac{\delta S[g]}{\delta g_{\mu\nu}} - \mu G_{\mu\nu\rho\sigma}\frac{\delta S[g]}{\delta g_{\mu\nu}}\frac{\delta S[g]}{\delta g_{\rho\sigma}}\right) - \mathcal{A}(g). \tag{3.51}$$

First, we require the cancellation of the term linear in the momentum. This gives us the following gradient condition:

$$\beta_{\mu\nu}(g) = 2\mu G_{\mu\nu\rho\sigma}\frac{\delta S[g]}{\delta g_{\rho\sigma}}. \tag{3.52}$$

The first term on the left hand side cancels against the second term on the right hand side. Then, we also see that the second term on the left hand side reads

$$\beta_{\mu\nu}\frac{\delta S[g]}{\delta g_{\mu\nu}} = 2\mu G_{\mu\nu\rho\sigma}\frac{\delta S[g]}{\delta g_{\mu\nu}}\frac{\delta S[g]}{\delta g_{\rho\sigma}}. \tag{3.53}$$

And, the flow equation then reduces to

$$\int \mathrm{d}^4x\sqrt{g}\left(-\frac{\mu}{\sqrt{g}}G_{\mu\nu\rho\sigma}\pi^{\mu\nu}\pi^{\rho\sigma}+\mu G_{\mu\nu\rho\sigma}\frac{\delta S[g]}{\delta g_{\mu\nu}}\frac{\delta S[g]}{\delta g_{\rho\sigma}}\right)-\mathcal{A}(g)=0. \tag{3.54}$$

The integrated Weyl anomaly for four dimensional conformal field theories takes the form:

$$\mathcal{A}[g]=\int \mathrm{d}^4x\sqrt{g}\left(\frac{c}{3}-a\right)R^2+(-2c+4a)R^{\mu\nu}R_{\mu\nu}+(c-a)R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}, \tag{3.55}$$

Then, recalling that the theory we are dealing with is holographically dual to general relativity, requires $a=c$, as discovered in [85]. So, we have

$$\mathcal{A}[g]=\int \mathrm{d}^4x\sqrt{g}2a\left(R^{\mu\nu}R_{\mu\nu}-\frac{1}{3}R^2\right)\equiv aA^{(a=c)}[g]. \tag{3.56}$$

For the function $S[g]$, if we take the Holographic counterterm of [14], i.e.[3]

$$S[g]=\frac{3}{\ell}\left(\int \mathrm{d}^4x\sqrt{g}\left(1+\frac{\ell^2R}{6}\right)\right), \tag{3.57}$$

then we find that first, the anomaly term cancels against the following term:

$$A^{(a=c)}=\int \mathrm{d}^4x\sqrt{g}G_{\mu\nu\rho\sigma}\frac{\delta\left(\int \mathrm{d}^4x\sqrt{g}R\right)}{\delta g_{\mu\nu}}\frac{\delta\left(\int \mathrm{d}^4x\sqrt{g}R\right)}{\delta g_{\rho\sigma}}, \tag{3.58}$$

if the identification $\frac{\mu\ell^2}{4}=a$ is made. This form of the $a=c$ anomaly also appeared in [129]. Recalling that $a=\frac{\pi^2\ell^3}{8G}$, for supergravity in $\mathrm{AdS}_5\times S^5$ which is dual to $\mathcal{N}=4$ Super Yang-Mills theory at large N, the relation found here implies that $\mu=\frac{\pi^2\ell}{2G}$.

A caveat is in order at this point: note that the correct counter-term to use at asymptotic infinity is the one presented in [20] that involves in addition to what is above a logarithmically divergent (as the asymptotic boundary is approached) term that multiplies a fourth derivative term arising from the anomaly itself. Thus an additional canonical transformation that involves the addition of that counter term must be performed before taking the asymptotic limit of quantities defined in the setup considered here in order to avoid divergences that would otherwise arise in the action on-shell.

[3] Again, this is just a local function that has the same form as the Holographic couterterm, but it isn't added with the intention to cancel any divergences, because at a finite radius boundary, there aren't any.

Then, the remaining terms in (3.54) organise themselves into

$$H = -\int \mathrm{d}^4x \left(\frac{G_{\mu\nu\rho\sigma}\pi^{\mu\nu}\pi^{\rho\sigma}}{\sqrt{g}} - \sqrt{g}\left(R + \frac{12}{\ell^2}\right)\right) = 0. \tag{3.59}$$

This is nothing but the Hamiltonian constraint of five dimensional general relativity with a negative cosmological constant, smeared against unit lapse. We also have the vector constraint density:

$$\nabla_\mu \pi^\mu_\nu = 0, \tag{3.60}$$

that follows from the covariant conservation of the stress tensor, and can be integrated against an arbitrary shift vector field to obtain the constraint.

The flow function now reveals that the boundary metric satisfies a generalized Ricci flow equation:

$$\beta_{\mu\nu}(g) = \mu\left(\frac{1}{\ell}g_{\mu\nu} + \frac{\ell}{2}\left(R_{\mu\nu} - \frac{1}{6}Rg_{\mu\nu}\right)\right). \tag{3.61}$$

This is similar to what was found in [94] and [123].

3.4.2 Including the scalar matter

In this section, the effect of adding a double trace scalar operator deformation to the above setup is considered. The effect of adding this deformation in the bulk is similar to that of the stress tensor double trace deformation in that it regulates the theory on the boundary in a way that corresponds to pulling the boundary to a finite radius.

First, note that the source $\phi(x)$ of the single trace scalar deformation is also space dependent and hence it contributes towards the anomaly, and for holographic theories it reads:[4]

$$\mathcal{A}[g,\phi] = \int \mathrm{d}^4x\sqrt{g}\Big(\alpha(\phi)R^{\mu\nu}R_{\mu\nu} - \gamma(\phi)R^2 + \zeta(\phi)\partial_\mu\phi\partial^\mu R + \eta(\phi)R(\partial^\mu\phi\partial_\mu\phi) +$$
$$+ \xi(\phi)\partial^\mu\phi\partial^\nu\phi\left(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\right) + \lambda(\phi)\nabla^2\phi\nabla^2\phi + \chi(g,\phi)_{\mu\nu\rho\sigma}\partial^\mu\phi\partial^\nu\phi\partial^\rho\phi\partial^\sigma\phi\Big). \tag{3.62}$$

Then the various functions are found in terms of those that go into the definition of the function $S[g,\phi]$ that specifies the canonical transformation relating the one-point function of $\mathcal{O}$ to the momentum p_ϕ conjugate to the source ϕ in the bulk Hamiltonian.

[4] See [?] the form of the scalar source and metric dependent anomaly in general CFTs

Recall that the flow equation now looks like:

$$
-\int \mathrm{d}^4x\sqrt{g}\left(\beta_{\mu\nu}(g,\phi)\frac{\delta}{\delta g_{\mu\nu}}+\beta_\phi(g,\phi)\frac{\delta}{\delta\phi}\right)\ln Z[g,\phi]=
$$

$$
=-\int \mathrm{d}^4x\sqrt{g}\left(\mu G_{\mu\nu\rho\sigma}\langle T^{\mu\nu}\rangle\langle T^{\rho\sigma}\rangle+\frac{\kappa}{2}\langle\mathcal{O}\rangle^2\right)+\mathcal{A}[g,\phi]. \tag{3.63}
$$

Like in the pure gravity case, we make the identification

$$
\sqrt{g}\langle\mathcal{O}\rangle = p_\phi-\frac{\delta S[g,\phi]}{\delta\phi}, \tag{3.64}
$$

in addition to the identification made of the momentum conjugate to the metric made in the previous section.

The flow equation then becomes

$$
\int \mathrm{d}^4x\left(\beta_{\mu\nu}(g,\phi)\left(\pi^{\mu\nu}-\frac{\delta S[g,\phi]}{\delta g_{\mu\nu}}\right)+\beta_\phi(g,\phi)\left(p_\phi-\frac{\delta S[g,\phi]}{\delta\phi}\right)\right)=
$$

$$
\int \mathrm{d}^4x\left(-\mu\frac{G_{\mu\nu\rho\sigma}}{\sqrt{g}}\left(\pi^{\mu\nu}-\frac{\delta S[g,\phi]}{\delta g_{\mu\nu}}\right)\left(\pi^{\rho\sigma}-\frac{\delta S[g,\phi]}{\delta g_{\rho\sigma}}\right)-\frac{\kappa}{2\sqrt{g}}\left(p_\phi-\frac{\delta S[g,\phi]}{\delta\phi}\right)^2\right)-\mathcal{A}[g,\phi]. \tag{3.65}
$$

then, like in the gravity case, we set:

$$
\beta_\phi(g,\phi)=\kappa\frac{\delta S[g,\phi]}{\delta\phi}. \tag{3.66}
$$

Then, the final expression for the bulk Hamiltonian reads

$$
\int \mathrm{d}^4x\left(-\mu\frac{G_{\mu\nu\rho\sigma}}{\sqrt{g}}\pi^{\mu\nu}\pi^{\rho\sigma}-\kappa\frac{p_\phi^2}{2\sqrt{g}}\right)=
$$

$$
-\mathcal{A}[g,\phi]+\int \mathrm{d}^4x\left(\mu G_{\mu\nu\rho\sigma}\frac{\delta S[g,\phi]}{\delta g_{\mu\nu}}\frac{\delta S[g,\phi]}{\delta g_{\rho\sigma}}+\frac{\kappa}{2}\left(\frac{\delta S[g,\phi]}{\delta\phi}\right)^2\right). \tag{3.67}
$$

Given that the scalar field (like any matter field) couples to gravity with a strength set by Newton's constant which has been set to unity, and that there is no other scale involved in this coupling requires $\mu=\kappa$. Then we divide throughout by μ and absorb the remaining factor of it in the denominator of the term involving $\mathcal{A}[g,\phi]$ by redefining the as yet undetermined functions in its definition. This equality is significant, as it implies that there is just one scale associated to the total double trace deformation. More will be said about this condition in the discussion.

In order for the momentum independent part of the above constraint equation to be of the form $\sqrt{g}\left(-\frac{(\partial_\mu\phi\partial^\mu\phi)}{2}+R+V(\phi)\right)$, the potential $S[g,\phi]$ can have two derivatives of the sources at most.

The ansatz made is

$$S[g,\phi]=\int \mathrm{d}^4x\sqrt{g}\left(X(\phi)+U(\phi)R+P(\phi)(\partial^\mu\phi\partial_\mu\phi)\right). \tag{3.68}$$

If we require the cancellation between the anomaly and the square of the derivatives of this functional present in the momentum independent terms of (3.67), then the flow equation now reads as the Hamiltonian constraint for the five dimensional bulk gravity- scalar system:

$$\int \mathrm{d}^4x\left\{-\frac{1}{\sqrt{g}}\left(G_{\mu\nu\rho\sigma}\pi^{\mu\nu}\pi^{\rho\sigma}+\frac{p_\phi^2}{2}\right)-\sqrt{g}\left(\frac{1}{2}\partial_\mu\phi\partial^\mu\phi+R+V(\phi)\right)\right\}=0, \tag{3.69}$$

provided the following superpotential like relations are satisfied:

$$\left(\frac{X'^2}{2}-\frac{X^2}{3}\right)=V,\quad \left(U'X'-\frac{UX}{3}\right)=1,\quad (PX')'-PX=1. \tag{3.70}$$

As promised before, the functions in the anomaly are related to the functions in the defintion of S, i.e.

$$\chi(g,\phi)_{\mu\nu\rho\sigma}=P^2\left(g_{\mu(\rho}g_{\sigma)\nu}-g_{\mu\nu}g_{\rho\sigma}\right),\alpha(\phi)=\frac{U^2}{2} \tag{3.71}$$

$$\gamma(\phi)=\left(\frac{U^2}{3}-\frac{U'^2}{2}\right),\quad \zeta(\phi)=PU' \tag{3.72}$$

$$\eta(\phi)=(PU')',\quad \xi(\phi)=PU,\quad \lambda(\phi)=\frac{P^2}{2}. \tag{3.73}$$

This analysis reproduces a part of the results obtained in [129] where a completely generic scalar coupled to AdS gravity was first considered. In all these expressions, the prime denotes partial differentiation with respect to ϕ, i.e. $()'\equiv\partial_\phi()$. Note that in deriving these conditions, many integrations by parts have been carried out and boundary terms have been discarded.

Some of these relations are identical to those derived previously in the context of holographic renormalization (in say, [98]). The flow function for the scalar field can also be interpreted as the beta function for the renormalization group flow triggered by the addition of the deformation. To leading order in perturbation theory, the beta function takes the form

$$\beta_\phi(g,\phi) \propto (4-\Delta)\phi + \cdots , \tag{3.74}$$

where Δ is the conformal dimension of the operator $\mathcal{O}$. This implies that the function $X(\phi)$ is given to leading order by:

$$X(\phi) = -\frac{6}{\ell} - \frac{1}{2\ell}(4-\Delta)\phi^2 + \cdots \tag{3.75}$$

The reason for the specific numerical factor in the ϕ independent part and the factors of ℓ being where they are is to ensure that the bulk potential computed through the relation:

$$V = \frac{X'^2}{2} - \frac{X^2}{3} = \frac{12}{\ell^2} - \frac{1}{2}\frac{\Delta(\Delta-4)\phi^2}{\ell^2} + \cdots \tag{3.76}$$

becomes the appropriate cosmological constant in the pure gravity Hamiltonian constraint when $\phi \to 0$.

Also, note that the mass-conformal dimension relationship

$$m^2 = \frac{\Delta(\Delta-4)}{\ell^2}, \tag{3.77}$$

has been recovered. So, for example, when $\Delta = 4$, i.e. when $\mathcal{O}$ is marginal, the bulk scalar field is massless and minimally coupled.

The expression of diffeomorphism invariance tangential to the hypersurfaces in this coupled system, takes the form:

$$\nabla_\nu \pi^{\mu\nu} + p_\phi \nabla^\mu \phi = 0, \tag{3.78}$$

which is the vector constraint in the bulk, whose form follows again from the Ward identity associated to diffeomorphism invariance of the holographic theory coupled to the metric and scalar sources.

3.4.3 Closure and Cancellation

Now we see that the key feature of the holographic anomaly which allows us to convert the flow equation (3.48) into the constraint equation (3.31) is the following property:

$$\mathcal{A}[g,\phi] - \int \mathrm{d}^4x \left(G_{\mu\nu\rho\sigma} \frac{\delta S[g,\phi]}{\delta g_{\mu\nu}} \frac{\delta S[g,\phi]}{\delta g_{\rho\sigma}} + \frac{1}{2}\left(\frac{\delta S[g,\phi]}{\delta \phi}\right)^2 \right) =$$

$$= \int \mathrm{d}^4x\sqrt{g}\left(-\frac{1}{2}\left(\partial^\mu\phi\partial_\mu\phi\right) + R + V\right). \tag{3.79}$$

This holds, provided the relations (3.70)-(3.73) all hold. In fact, these relations were derived above by requiring (3.79) to hold, which in turn came from wanting to transform the flow equation (3.48) into the constraint (3.31).

We could however have chosen to derive this relation from an alternative, yet equivalent demand. Namely, starting from (3.48) and making the appropriate change of variables, we are led to (3.67), which we reproduce here:

$$H = \int \mathrm{d}^4x N(x)\Bigg\{ -\mu\frac{G_{\mu\nu\rho\sigma}}{\sqrt{g}}\pi^{\mu\nu}\pi^{\rho\sigma} - \kappa\frac{p_\phi^2}{2\sqrt{g}} +$$

$$+\left(\mathcal{A}[g,\phi] - \mu G_{\mu\nu\rho\sigma}\frac{\delta S[g,\phi]}{\delta g_{\mu\nu}}\frac{\delta S[g,\phi]}{\delta g_{\rho\sigma}} - \frac{\kappa}{2}\left(\frac{\delta S[g,\phi]}{\delta\phi}\right)^2\right)\Bigg\} = 0. \tag{3.80}$$

Note that independently of this equation, we are granted the vector constraint (3.78), which simply follows form the diffeomorphism Ward identity of the energy momentum tensor and the scalar source.

Without assuming that the final expression for the scalar constraint should take any particular form, but simply that its Poisson algebra closes, i.e. that

$$\{H(N), H(M)\} \approx 0, \tag{3.81}$$

where the symbol $\approx$ here denotes equality on the sub-space of phase space where the constraints are satisfied [5], implies that the relation (3.79) must hold. This is guaranteed by a theorem proven in [56] and strengthened further in [**?**]. The assumptions that go behind the theorem are simply that the vector constraint holds and that the momentum dependent part of the scalar constraint is quadratic and ultralocal. Although it was proven there for the pure gravity case, it is not hard to see that for the case at hand, where the scalar field's kinetic term in the Hamiltonian takes the form $\frac{p_\phi^2}{2\sqrt{g}}$, the potential term has to contain a derivative independent potential $V(\phi)$ in addition to the term $-\frac{(\partial_\mu\phi\partial^\mu\phi)}{2}$ in order to close at all.

Note that the closure condition (3.81) is in fact weaker than the statement of the Poisson algebra. There, the specific combination of constraints that vanish on the right-hand side of

[5]The right-hand side of (3.81) is, in general, some combination of the constraints $H(N)$ and the vector constraint, and so it vanishes when they are satisfied. In the language of constrained Hamiltonian dynamics, satisfying this closure property qualifies constraints to be 'first-class'.

(3.81) and the corresponding structure functions (as opposed to constants due to the field dependence) are also known. Thus, the condition (3.81) is not quite the same as demanding the emergence of bulk diffeomorphism invariance which is what is encoded in the Poisson algebra. It is the non-linear generalization of the requirement that five-dimensional general relativity only propagates the spin two modes of the graviton in addition to the coupled scalar field.

That being said, the uniqueness of the form of the constraint functions that satisfy (3.81) given (3.78) implies that the only way to satisfy the closure condition is in the manner general relativity does. In other words, there is something unique about how the diffeomorphism invariance tangential to an embedded hypersurface gets promoted to the full diffeomorphism invariance of the ambient spacetime, at least at the level of the constraints that generate the corresponding transformations on phase space.

The requirement that the momentum dependent 'kinetic term' in the constraint equation be quadratic and ultralocal, i.e. of the form

$$-\frac{G_{\mu\nu\rho\sigma}\pi^{\mu\nu}\pi^{\rho\sigma}}{\sqrt{g}}-\frac{p_\phi^2}{\sqrt{g}}, \tag{3.82}$$

is equivalent to the statement that the radial velocities in the bulk of the fields $g_{\mu\nu}$ and ϕ are at most linear in the momenta. In other words:

$$\sqrt{g}\left(\beta_{\mu\nu}(g,\phi)-\nabla_{(\mu}\xi_{\nu)}\right)=\pi_{\mu\nu}-\frac{1}{2}g_{\mu\nu}\pi^\rho_\rho, \quad \sqrt{g}(\beta_\phi(g,\phi)-\xi^\nu\nabla_\nu\phi)=p_\phi. \tag{3.83}$$

This feature of the flow was also noted in [80], and follows directly from the structure of the double trace deformations being added, and is not sensitive to the background fields organizing themselves into any particular form, which is what (3.79) reflects. We see here however that these two features are intimately related through the closure condition.

In the rest of this chapter, $\beta_\phi(g,\phi),\beta_{\mu\nu}(g,\phi)$ are interpreted not just as beta functions for the holographic quantum field theory, but also as the flow functions of the gradient flow regularization applied to the induced gravity theory obtained from integrating out the fields of this quantum field theory.

3.5 Generalized gradient flows and holography

One way to think of the holographic duality is that it operates between quantum field theories in D dimensions and gravitational theories in $D+1$ dimensions. However, when arbitrary background sources are turned on in the quantum field theory, there is an alternative statement of the duality which is seemingly more mundane but perhaps still illuminating. First, one interprets the generating functional of the quantum field theory in the presence of said sources, metric included, as a non-local induced gravity theory. Then, the holographic duality implies that this D dimensional non-local induced gravity theory and the $D+1$ dimensional theory, i.e. general relativity coupled to matter fields with negative cosmological constant, are equivalent to each other. This way of thinking about the duality was emphasized in [107].

This section aims to explain how this duality manifests when the boundary is at a finite radius. The key mechanism behind this version of the duality will be the generalized gradient flow, which will be discussed briefly in what follows.

The gradient flow ([110], [8]) is a method to regulate the correlation functions of composite operators of various quantum field theories in the coincidence limit. The idea is to append an additional dimension to the space on which the quantum field theory lives, and declare that the dependence of the fields of the theory on this additional dimension are dictated by gradient flow conditions. It was introduced in the context of Yang-Mills theory but can be applied to more general quantum field theories as well.

3.5.1 How gradient flows lead to smearing

To illustrate the idea, consider the simple example of applying it to the $O(N)$ non-linear sigma model in two dimensions.[6]. The action for the theory reads

$$S[\sigma] = \frac{1}{2g^2} \int \mathrm{d}^2x \left(h_{ab}(\sigma) \left(\partial_\mu \sigma^a \partial^\mu \sigma^b \right) \right), \tag{3.84}$$

where the fields $\sigma^a(x)$ are multi component scalars and the metric $h_{ab}(\sigma)$ is given by the inverse of

$$h^{ab}(\sigma) = \delta^{ab} - \sigma^a \sigma^b. \tag{3.85}$$

The generalized gradient flow method for regularizing the divergences in correlators of composite operators in the contact limit starts by appending an additional dimension to the two dimensional space on which the field theory lives with an additional scale dimension λ: $\sigma^a(x) \to \sigma^a(x, \lambda)$. Then, the dependence of the fields along this dimension is dictated by the flow equation

$$\frac{\partial \sigma^a(x,\lambda)}{\partial \lambda} = -h^{ab}(\sigma(x,\lambda)) \frac{\delta S[\sigma(x,\lambda)]}{\delta \sigma^b} |_{\sigma^a(x) \to \sigma^a(x,\lambda)}. \tag{3.86}$$

[6]The explanations here closely follow those in [8]

The $|_{\sigma^a(x)\to\sigma^a(x,\lambda)}$ denotes the boundary condition $\sigma^a(x,\lambda=0)=\sigma^a(x)$ at the boundary of the three dimensional space. The potential driving this flow, $S[\sigma(x,\lambda)]$ is the non-linear sigma model action where the fields $\sigma^a(x)$ are extended to the 'flowed' fields $\sigma^a(x,\lambda)$.

Explicitly, the flow equation reads:

$$\frac{\partial\sigma^a(x,\lambda)}{\partial\lambda}=\partial_\mu\partial^\mu\sigma^a+\sigma^a(\partial_\mu\sigma^b\partial^\mu\sigma_b)+\frac{\sigma^a(\partial^\mu\sigma^b)(\partial_\mu\sigma_b)}{4(1-\sigma^c\sigma_c)}, \tag{3.87}$$

where the indices of the internal vector components are contracted with the metric $h_{ab}(\sigma)$. For simplicity, consider the linearization of the above equation. This takes the form of a heat equation with the role of time being played by the flow parameter λ. The solution to the linearized equation then implies that to leading order in λ, the dependence of $\sigma^a(x,\lambda)$ on the flow time is given by smearing the original fields $\sigma^a(x)$ with the heat kernel:

$$\sigma^a(x,\lambda)=\exp\left(\lambda\left(\partial^2\right)\right)\sigma^a(x)+\cdots \tag{3.88}$$

The non locality associated to this smearing is what renders the correlators of composite operators built out of $\sigma^a(x,\lambda)$ finite.

3.5.2 Gradient flows for the induced boundary gravity theory

The similarity between this general procedure and the holographic duality leads one to wonder whether they coincide when applied to theories expected to possess holographic duals. This was the focus of [5], [6], [9], [10] and [7], where the generalized gradient flow is applied to the $O(N)$ vector model, and various aspects of the holographic duality having to do with reconstructing the bulk metric, understanding the effects of diffeomorphisms in the bulk and even computing $1/N$ corrections to the cosmological constant were considered.

Here the aim is more modest, unlike the work mentioned in the previous paragraph which aims to understand the duality constructively through implementing the gradient flow, here we just notice there are gradient flows hiding in holography at a finite radius as implemented through the double trace deformations described in the previous sections. First, it would help to identify which theory this procedure is being applied to. The equations

$$\frac{1}{\mu}\frac{\partial g_{\mu\nu}}{\partial\lambda}=G_{\mu\nu\rho\sigma}\frac{\delta S[g,\phi]}{\delta g_{\rho\sigma}},\quad \frac{1}{\mu}\frac{\partial\phi}{\partial\lambda}=\frac{\delta S[\phi,g]}{\delta\phi}, \tag{3.89}$$

certainly seem to have the structure of the generalized gradient flow equations barring the fact that they are not describing the flow of the fundamental fields of the theory itself but of its sources. This motivated Jackson et. al. in [94] to call equations such as these 'geometric RG flow' equations. However, if one considers the induced gravity theory in four dimensions that arises from integrating out the fundamental fields of the CFT. It is obtained by computing the generating functional as a function of the sources:

$$\ln Z[g,\phi] = \mathcal{S}^{ind}[g,\phi], \tag{3.90}$$

and interpret the resulting, non-local function of the metric and the other sources as the effective action for a four-dimensional gravitational theory. The holographic duality appears to really coincide with applying the gradient flow regularization to this induced gravity theory, where the dependence of its fields $g_{\mu\nu}$ and ϕ on the additional dimension parameterized by λ is given by the equations (3.52) and (3.66).

The statement of the holographic correspondence in terms of this induced theory is fairly straightforward at least at large N. It is just the statement that the effective action $\mathcal{S}^{ind}[g,\phi]$ satisfies the Hamilton–Jacobi equations of general relativity in one higher dimension:

$$\frac{\partial \mathcal{S}^{ind}[g,\phi]}{\partial\lambda} = -\frac{1}{\sqrt{g}}\left(G_{\mu\nu\rho\sigma}\frac{\delta\mathcal{S}^{ind}}{\delta g_{\mu\nu}}\frac{\delta\mathcal{S}^{ind}}{\delta g_{\rho\sigma}} + \frac{1}{2}\left(\frac{\delta\mathcal{S}^{ind}}{\delta\phi}\right)^2\right) - \sqrt{g}\left(-\frac{(\partial_\mu\phi\partial^\mu\phi)}{2} + R + V(\phi)\right) = 0, \tag{3.91}$$

and therefore should be identified with the on shell action for the bulk theory. This was noticed first by Liu and Tseytlin in [107], where checks at the level of the linearized theory in the bulk were performed. The only novel insight here has to do with the radial development of the induced theory leading to gradient flow conditions for the theory's fundamental fields.

3.6 Getting from the local CS equation to the Hamiltonian constraint in $d = 3, 4, 5$

We've mostly concentrated in the previous sections on boundary dimensions 2 and 5. However, even in dimensions between 2 and 5, the large N local CS equation coming from the deformation of CFTs by the T^2 deformation can be mapped to the Hamiltonian constraint equation in one higher dimension.

In dimensions higher than 2, the deforming operator $\mathcal{O}(x)$ is defined as

$$\mathcal{O}(x) = \lim_{y\to x}\frac{1}{4}\left(T_{\mu\nu}(x) - \frac{1}{D-1}T^{\kappa}_{\kappa}(x)g_{\mu\nu}(x)\right)T^{\mu\nu}(y). \tag{3.92}$$

It helps to introduce the de Witt super metric in any number of dimensions:

$$G_{\mu\nu\rho\sigma}(x) = \left(g_{\mu(\rho}(x) g_{\sigma)\nu}(x) - \frac{1}{D-1} g_{\mu\nu}(x) g_{\rho\sigma}(x) \right), \tag{3.93}$$

so that

$$T_{\mu\nu}(x) - \frac{1}{D-1} T^{\kappa}_{\kappa}(x) g_{\mu\nu}(x) = G_{\mu\nu\rho\sigma}(x) T^{\rho\sigma}(x). \tag{3.94}$$

From the definition of the energy momentum tensor, we have

$$\langle \mathcal{O}(x) \rangle Z[g] = \lim_{y\to x} G_{\mu\nu\rho\sigma}(x) \left(\frac{1}{\sqrt{g(x)}} \frac{\delta}{\delta g_{\mu\nu}(x)} \left(\frac{1}{\sqrt{g(y)}} \frac{\delta Z[g]}{\delta g_{\rho\sigma}(y)} \right) \right). \tag{3.95}$$

In order to generate the R term in the Hamiltonian constraint, we will implement the coincidence limit through the heat kernel. This method is similar to the one of [93] although the context is quite different. The heat kernel $K(x, y; \epsilon)$, satisfies the property

$$\lim_{\epsilon\to 0} K(x, y; \epsilon) = \delta(x, y). \tag{3.96}$$

This property should be thought of as an initial condition for the heat equation

$$\partial_\epsilon K(x, y; \epsilon) = (\nabla^2_{(x)} + \xi R_{(x)}) K(x, y; \epsilon). \tag{3.97}$$

We can now implement the point splitting regularization as follows

$$\lim_{y\to x} G_{\mu\nu\rho\sigma} \langle T^{\mu\nu}(x) T^{\rho\sigma}(y) \rangle Z[g] = \lim_{\epsilon\to 0} \int \mathrm{d}^D y K(x, y; \epsilon) G_{\mu\nu\rho\sigma}(x) \frac{1}{\sqrt{g}(x)} \frac{\delta}{\delta g_{\mu\nu}(x)} \left(\frac{1}{\sqrt{g}(y)} \frac{\delta Z[g]}{\delta g_{\rho\sigma}(y)} \right). \tag{3.98}$$

We also exploit the fact that we can add to the effective action terms involving local functions of the metric

$$Z[g] \to e^{C[g]} Z[g], \tag{3.99}$$

where $C[g]$ is chosen to be

$$C[g] = \alpha_0 \left(\epsilon^{\frac{D}{2}-1} \int \mathrm{d}^D x \sqrt{\gamma} + \frac{(D^2-3)\epsilon^{\frac{D}{2}}}{D(D-1)} \int \mathrm{d}^D x \sqrt{\gamma} R \right). \tag{3.100}$$

Here, α_0 is a constant given by

$$\alpha_0 = \frac{\alpha_d}{\lambda^{\frac{D+2}{2}}} \left(\frac{D-2}{2D^2 \kappa(D)} \right), \tag{3.101}$$

where

$$\kappa(D) = \frac{(D^2-3)(D(D(9D-11)-28)+42)}{12D(D-1)^2}. \tag{3.102}$$

With this choice of ϵ scaling in the improvement term $C[g]$, one can show (as we do in appendix **??**) that the deforming operator becomes

$$\langle \mathcal{O}(x) \rangle = \lim_{\epsilon \to 0} \int \mathrm{d}^D y K(x,y,\epsilon) G_{\mu\nu\rho\sigma}(x) \langle T^{\mu\nu}(x) T^{\rho\sigma}(y) \rangle + \alpha_0 R(x) \tag{3.103}$$

which we then subject to the large N limit to obtain

$$\begin{aligned} \langle \mathcal{O}(x) \rangle|_{N\to\infty} &= \lim_{\epsilon\to 0} \textstyle\int \mathrm{d}^D y K(x,y;\epsilon) G_{\mu\nu\rho\sigma}(x) \langle T^{\mu\nu}(x) \rangle \langle T^{\rho\sigma}(y) \rangle + \alpha_0 R(x) & (3.104) \\ &= G_{\mu\nu\rho\sigma}(x) \langle T^{\mu\nu}(x) \rangle \langle T^{\rho\sigma}(x) \rangle + \alpha_0 R(x). & (3.105) \end{aligned}$$

Here we have used the fact that the large N factorized two point function does not suffer any coincidence divergences so the limit can be taken to turn the heat kernel into a delta function, and the y integral can be performed. We can plug this back into the local CS equation (**??**), which now reads, at large N

$$T^{\nu}_{\nu} = -D\lambda G_{\mu\nu\rho\sigma} T^{\mu\nu} T^{\rho\sigma} - \frac{(D-2)\alpha_D}{2\lambda^{\frac{D-2}{2}}} R - \mathcal{A}(\gamma). \tag{3.106}$$

In $d=3$ and $d=5$, the anomaly $\mathcal{A}(\gamma)=0$. Here we immediately obtain (**??**) provided we make the choice $\hat{T}^{ij} = T^{ij}$. In $d=4$, the holographic anomaly is given by

$$\mathcal{A} = -\frac{\alpha_4^2}{\lambda^2} \left(G_{\mu\nu} G^{\mu\nu} - \frac{1}{3} (G^{\nu}_{\nu})^2 \right), \tag{3.107}$$

where $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}$ is the Einstein tensor, and a is the anomaly coefficient. This can be absorbed into an improvement of the energy momentum tensor, which is subsumed in the definition of the bare energy momentum tensor

$$\hat{T}^{\mu\nu} = T^{\mu\nu} + \frac{\alpha_4}{\lambda} G^{\mu\nu}. \tag{3.108}$$

In other words, the equation

$$T^{\kappa}_{\kappa} = -4\lambda \left(T^{\mu\nu} T_{\mu\nu} - \frac{1}{D-1} (T^{\kappa}_{\kappa})^2\right) - \frac{\alpha_4}{\lambda} R - \mathcal{A}(\gamma)$$

becomes

$$T^{\kappa}_{\kappa} + \frac{\alpha_4}{\lambda} G^{\kappa}_{\kappa} = -4\lambda \left(\left(T^{\mu\nu} + \frac{\alpha_4}{\lambda} G^{\mu\nu} \right) \left(T_{\mu\nu} + \frac{\alpha_4}{\lambda} G_{\mu\nu} \right) - \frac{1}{d-1} \left(T^{\kappa}_{\kappa} + \frac{\alpha_4}{\lambda} G^{\kappa}_{\kappa} \right)^2 \right) - \frac{\alpha_4}{\lambda} R. \tag{3.109}$$

Limitations of this method

One can ask what justifies the specific choices such as the powers of ϵ appearing in the definition of $C[g]$, and the choice of $\xi(D)$ that appears in the appendix **??**. For now, we can only offer a post facto justification, in that these choices lead to the form of the flow equation. It would be interesting to find an intrinsically field-theoretic justification for this scheme.

Also, in odd dimensions, the information about the counterterms one can add to the energy-momentum tensor cannot be divined from transforming the flow equation to reveal an anomaly term, as one can in four dimensions.

Chapter 4

Entanglement Entropy and the $T\bar{T}$ deformation

In this chapter, we will pivot towards questions relating to quantum information theory and finite cutoff holography. In particular, we will describe the computation of the holographic entanglement entropy, which is both sensitive to the ultraviolet physics of the theory on the boundary and probes the bulk geometry. We will first describe how this quantity can be computed at large c, where the bulk geometry is classical and then we will describe how $1/c$ corrections can be accounted for. Such corrections will correspond to quantum corrections in the bulk.

To set the stage, we will introduce notions of entanglement entropy and the Hartle–Hawking state, which we will focus on in this chapter.

4.1 Entanglement entropy

Here is a brief review of entanglement entropy and its calculation in field theory. A nice introductory reference is chapter 3 of [145].

In quantum field theory, we can adopt the functional Schrodinger picture where we view the state as a functional of the fields at a given time. So a state is a wavefunctional $\psi[\Phi]$ where $\Phi(x) : x \in \Sigma$ is the field at on a given time slice Σ. We can denote the Hilbert space of all such functionals $\mathcal{H}_\Sigma$.

If we divide Σ into two disjoint pieces $\Sigma = A \cup B$, $A \cap B = \emptyset$ then we can consider restricting the field separately to A and B, and view the field as a pair of functions $\Phi =$

(Φ_A, Φ_B). The wavefunctional is now a functional of two arguments, $\psi(\Phi) = \psi(\Phi_A, \Phi_B)$. Thus we can view it as a state in a tensor product Hilbert space,[1]

$$\psi \in \mathcal{H}_\Sigma = \mathcal{H}_A \otimes \mathcal{H}_B. \tag{4.1}$$

Given such a state, we can consider the reduced density matrices

$$\rho_A =_{\mathcal{H}_B} (\psi\psi), \qquad \rho_B =_{\mathcal{H}_A} (\psi\psi). \tag{4.2}$$

Here ρ_A tells us all we need to know if we make measurements confined to region A.

To quantify the amount of entanglement between regions A and B we use entanglement entropy, defined as

$$S = -(\rho_A \log \rho_A) = -(\rho_B \log \rho_B). \tag{4.3}$$

More generally, given a reduced density matrix ρ_A, we define the *modular Hamiltonian* $H_A = -\log(\rho_A)$. The spectrum of H_A is called the *entanglement spectrum* and contains a lot more information than just the entanglement entropy. Note that as we have defined it H_A is dimensionless, unlike a usual Hamiltonian which has dimensions of energy.

Unfortunately, we hardly ever calculate entanglement entropy this way. The issue is that to calculate $\log(\rho_A)$ we should first diagonalize ρ_A, and in general this is hard to do.[2] Instead, we use the following trick.

4.1.1 The Hartle-Hawking state

The main method used to calculate entanglement entropy in field theory uses the Euclidean path integral. Let us consider a state that is prepared by the Euclidean path integral. The simplest example is the ground state, but we will be interested in the *Hartle-Hawking state* in 1+1 de Sitter space [95].

Recall that the metric of 1+1 de Sitter space dS_2 is

$$ds^2 = r^2(-d\tau^2 + \cosh(\tau)^2 d\phi^2). \tag{4.4}$$

[1] We have skimmed over a lot of details here, really one should be very careful about exactly which functionals one is allowing, etc. When we do this, we find that the tensor product decomposition is not quite true, basically, the states in $\mathcal{H}_A$ and $\mathcal{H}_B$ are too singular to really exist. We can make all of this precise by introducing a regulator such as a lattice, but the price we pay is that the quantities we wish to define will diverge when we take the lattice spacing to zero. Our strategy will be to boldly forge ahead ignoring these issues, but we should be careful in interpreting the results we get.

[2] However, there are exceptions: see [144], where an explicit calculation is done for a free scalar field with a lattice regulator.

To define the Hartle-Hawking state we Wick rotate $\tau = i\theta$ and we find

$$ds^2 = r^2(d\theta^2 + \cos(\theta)^2 d\phi^2) \tag{4.5}$$

which we recognize as the metric of the 2-sphere of radius r.

The Hartle-Hawking state is defined as a wave functional on the $\theta = 0$ slice of dS_2. We integrate over all field configurations in the "lower hemisphere" $\theta < 0$ with fixed boundary conditions on the equator:

$$\psi_{\mathrm{HH}}[\Phi_0] = \int_{\Phi(\theta=0,\phi)=\Phi_0(\phi)} [\mathcal{D}\Phi]\, e^{-I[\Phi]}. \tag{4.6}$$

Here we're using $I[\phi]$ for the Euclidean action since we already used S for the entropy.

To get a Hamiltonian description of this process, we can slice up the path integral; this is just the usual way we define the path integral from the Hamiltonian formulation but in reverse. We will be interested in the entanglement between two halves of the equator. It will be useful to rotate our coordinate system so that the boundary separating systems A and B consists of the two poles at $\theta = \pm\pi/2$. Let K be the generator of rotations in ϕ, which fixes those two points. By inserting resolutions of identity along the angular slicing we find

$$\psi_{\mathrm{HH}}(\phi_A, \phi_B) = \langle \phi_A | e^{-\pi K} | \phi_B \rangle. \tag{4.7}$$

In the original Lorentzian spacetime, K generates "de Sitter boosts" which are symmetries that preserve a single observer's static patch.

Now to obtain the reduced density matrix, we put two path integrals together and trace along the opposite region. The result is:

$$\langle \phi'_A | \rho_A | \phi_A \rangle = \langle \phi'_A | e^{-2\pi K} | \phi_A \rangle. \tag{4.8}$$

In other words, the modular Hamiltonian is $H_A = 2\pi K$. So the entanglement spectrum can be determined from the spectrum of the de Sitter boost K.

4.1.2 The replica trick

To find the entropy, it is useful to be able to vary the temperature. In our case, the "energy" H_A is dimensionless, and so the temperature is too. The partition function at temperature $1/n$ is

$$Z = e^{-nH_A} \tag{4.9}$$

which corresponds to a Euclidean path integral on a sphere where the range of the azimuthal angle is changed from $[0, 2\pi)$ to $[0, 2\pi n)$. The resulting manifold is not smooth, it has conical singularities at two points.

Knowing Z as a function of n is equivalent to knowing the full entanglement spectrum; (4.9) is the Laplace transform of the entanglement spectrum, so we "just" have to invert the Laplace transform.

To extract the entropy at temperature $1/n$, we use an identity from thermodynamics:

$$S = \left(1 - n\frac{d}{dn}\right) \log Z. \tag{4.10}$$

To get the entanglement entropy, we evaluate this expression at $n = 1$.

4.1.3 The sphere trick

The replica trick is useful, but it requires us to consider manifolds with conical singularities. In the de Sitter case, there is a further simplification that makes use of the rotational symmetry [49]. Unfortunately, it only gives us the entanglement entropy, not the full spectrum.

At $n = 1$, the entropy is given by

$$S = \log Z - \langle K \rangle. \tag{4.11}$$

The generator K is given in terms of the stress tensor by

$$K = r^2 \int_{-\pi/2}^{\pi/2} d\theta \cos(\theta) T^{\phi}_{\phi}. \tag{4.12}$$

And the expectation value is taken in the Hartle-Hawking state of the sphere.

Now we make use of the spherical symmetry: the stress tensor must be proportional to the metric $\langle T_{\mu\nu} \rangle = \alpha g_{\mu\nu}$. This means we can relate K directly to the trace of the stress tensor, $T^{\phi}_{\phi} = \frac{1}{2} T^{\mu}_{\mu}$, and hence $\langle K \rangle = \frac{1}{2} \int d^2x \sqrt{g} \langle T^{\mu}_{\mu} \rangle$. Using the definition of the stress tensor, we find

$$S = \left(1 - \frac{r}{2}\frac{d}{dr}\right) \log Z. \tag{4.13}$$

Thus to find the entanglement entropy of the Hartle-Hawking state, we only need to know the partition function of the sphere, which is a huge simplification.

Now, we can apply the sphere trick to the $T\bar{T}$ deformed CFT at large c.

4.2 Large c von Neumann and Conical Entropy

We first calculate the sphere partition function in the $T\overline{T}$-deformed CFT. We will see that this is sufficient to calculate the entanglement entropy when the entangling surface is two antipodal points on the sphere.

To find the sphere partition function, we consider the metric $ds^2 = r^2(d\theta^2 + \sin(\theta)^2 d\phi^2)$ and vary the radius r:

$$\frac{d}{dr}\log Z = -\frac{1}{r}\int d^2x\sqrt{g}\,\langle T^\mu_\mu\rangle. \tag{4.14}$$

By symmetry, the stress tensor takes the form $T_{\mu\nu} = \alpha g_{\mu\nu}$, where α is determined by the large c flow equation:

$$\langle T^\mu_\mu\rangle = -\frac{c}{24\pi}R - \frac{\mu}{4}\left(\langle T^{\mu\nu}\rangle\langle T_{\mu\nu}\rangle - \langle T^\mu_\mu\rangle^2\right). \tag{4.15}$$

which becomes a quadratic equation for α. The solution is:

$$\alpha = \frac{2}{\mu}\left(1 - \sqrt{1 + \frac{c\mu}{24\pi r^2}}\right). \tag{4.16}$$

In solving the quadratic equation we have chosen the branch that gives the CFT trace anomaly in the limit $\mu \to 0$. This yields a differential equation for the partition function as a function of radius

$$\frac{d\log Z}{dr} = \frac{16\pi}{\mu}\left(\sqrt{r^2 + \frac{c\mu}{24\pi}} - r\right). \tag{4.17}$$

This can be integrated with the help of the substitution $r = \sqrt{\frac{c\mu}{24\pi}}\sinh(x)$, giving the sphere partition function

$$\log Z = \frac{c}{3}\left(x + \frac{1}{2}\left(1 - e^{-2x}\right)\right) \tag{4.18}$$

$$= \frac{c}{3}\sinh^{-1}\left(\sqrt{\frac{24\pi}{c\mu}}r\right) + \frac{8\pi}{\mu}\left(r\sqrt{\frac{c\mu}{24\pi} + r^2} - r^2\right) \tag{4.19}$$

Note that we have chosen the boundary condition $\log Z = 0$ at $r = 0$; this would not have been possible in a CFT, where the partition function continues to change as a function of scale at arbitrarily short distances. Here we see that the flow equation is consistent with a trivial theory in the UV.

The Euclidean path integral on S^2 corresponds to the de Sitter vacuum. We will consider the entanglement entropy of this state across an entangling surface consisting of two antipodal points. This entropy can be obtained directly from the sphere partition function as follows.[3] To calculate the entanglement entropy by the replica trick, we consider the n-sheeted cover of the sphere:

$$ds^2 = r^2(d\theta^2 + n^2 \sin(\theta)^2 d\phi^2). \tag{4.20}$$

The entropy is then obtained from the partition function as:

$$S = \left(1 - n\frac{d}{dn}\right) \log Z\Big|_{n=1}. \tag{4.21}$$

In the absence of rotational symmetry, this formula requires analytic continuation in n, but in the case of antipodal points we can continuously vary n.

Under a change of n, the partition function changes as

$$\frac{d\log Z}{dn}\Big|_{n=1} = -\int \sqrt{g}\, T^{\phi}_{\phi}. \tag{4.22}$$

Since the stress tensor on the sphere is isotropic, $T^{\phi}_{\phi} = \frac{1}{2}T^{a}_{a}$. From (4.14) we conclude that the entropy can be expressed in terms of the sphere partition function as

$$S = \left(1 - \frac{r}{2}\frac{d}{dr}\right) \log Z = \frac{c}{3}\sinh^{-1}\left(\sqrt{\frac{24\pi}{c\mu}}r\right). \tag{4.23}$$

For $r \gg \sqrt{\mu c}$ we see that this formula reproduces the well-known CFT result [87, 24] with subleading corrections

$$S = \frac{c}{3}\log\left(\sqrt{\frac{96\pi}{c\mu}}r\right) + \frac{c^2\mu}{288\pi r^2} + O\left(\mu^2\right). \tag{4.24}$$

The corrections to the logarithmic term in the entanglement entropy are polynomial in μ starting from order one. In the UV limit, $r \to 0$ the entanglement entropy vanishes, indicating that the theory flows to a trivial theory.

We can also compare this result with the holographic proposal of [114]. The metric of Euclidean AdS$_3$ in global coordinates is

$$ds^2 = \ell^2\big(d\rho^2 + \sinh(\rho)^2(d\theta^2 + \sin(\theta)^2 d\phi^2)\big). \tag{4.25}$$

[3] We thank Aron Wall for teaching us this trick.

The sphere of radius r embeds into this geometry as a surface $\rho = \rho_0$, where ρ_0 is defined by $r = \ell \sinh(\rho_0)$. According to the conjecture of [114], the $T\overline{T}$ deformed theory is dual to quantum gravity in the region $\rho < \rho_0$. According to the Ryu-Takayanagi formula [137], the holographic entanglement entropy is given by $L/(4G)$ where L is the length of a minimal geodesic connecting the points of the entangling surface. In the case of antipodal points, the geodesic passes through the center and has length $L = 2\ell\rho_0$. Thus the Ryu-Takayanagi formula yields:

$$S = \frac{L}{4G} = \frac{\ell}{2G} \sinh^{-1}\left(\frac{r}{\ell}\right), \tag{4.26}$$

which agrees precisely with (4.23) with the identifications (**??**).

4.2.1 Conical entropy

The entanglement entropy is just one measure of entanglement. More information about the spectrum of the reduced density matrix is encoded in the conical entropy [4]:

$$\tilde{S}_n = \left(1 - n\frac{d}{dn}\right) \log Z_n, \tag{4.28}$$

which reduces to the entanglement entropy when $n = 1$. In the case of two antipodal points on the sphere, $\tilde{S}_n$ is the de Sitter entropy at a temperature $\sim 1/n$. By varying n we probe the density of states at different energy scales.

To calculate $\tilde{S}_n$ via the replica trick, we consider the partition function of the theory on the conical sphere (4.20). We will assume rotational symmetry in ϕ, so that we can parametrize the stress tensor in terms of two nonzero components $T_{\theta\theta}(\theta)$ and $T_{\phi\phi}(\theta)$. The problem simplifies if we consider the variables $u = \frac{\mu}{2}T^\theta_\theta - 1$ and $v = \frac{\mu}{2}T^\phi_\phi - 1$, in terms of these variables stress tensor conservation and the flow equation are

$$\frac{du}{d\theta} = \cot(\theta)(v - u), \qquad uv = 1 + \frac{c\mu}{24\pi r^2}. \tag{4.29}$$

This has the solution

$$u^2 = 1 + \frac{c\mu}{24\pi r^2} + \frac{c_n}{\sin(\theta)^2}, \tag{4.30}$$

[4]The conical entropy $\tilde{S}_n$ is related to the Rényi entropy S_n as

$$\tilde{S}_n = n^2 \partial_n \left(\frac{n-1}{n} S_n\right). \tag{4.27}$$

where c_n is independent of θ. The value of c_n is determined by the coupling of the theory to the conical singularity at $\theta = 0, \pi$.

Boundary conditions We will define the theory on a surface with a conical singularity via a limiting procedure similar to Ref. [64]. We first define a family of smoothed replica geometries in which a small neighborhood of the conical singularity is replaced by a smooth "cap". We will take this cap to be a portion of maximally symmetric space; for $n < 1$ this is a sphere, and for $n > 1$ a hyperbolic space.

Near the conical singularity, the geometry is approximately a flat cone $ds^2 = d\tau^2 + \tau^2 n^2 d\phi^2$, from which we cut out the region $\tau < \epsilon$. For $n > 1$ we attach this to a cap which is the region $\sigma < \sigma_c$ of the hyperbolic space with metric $ds^2 = \ell_c^2(d\sigma^2 + \sinh(\sigma)^2 d\phi^2)$, which has constant negative curvature $R = -2/\ell_c^2$. Matching the intrinsic and extrinsic geometry of the circle determines

$$\ell_c = \frac{\epsilon n}{\sqrt{n^2 - 1}}, \qquad \sigma_c = \cosh^{-1}(n). \tag{4.31}$$

The nonsingular solution on the cap takes the form $T_{ab} = \frac{2}{\mu}(u + 1)$ where

$$u^2 = 1 + \frac{c\mu}{24\pi\epsilon^2}\left(\frac{1}{n^2} - 1\right). \tag{4.32}$$

However, (4.32) has no real solutions for small ϵ which prevents us from taking the limit $\epsilon \to 0$.[5] In terms of bulk variables, there are no solutions when $\ell_c < \ell$: this corresponds to the fact that we cannot embed a hypersurface with more negative curvature than the ambient space without breaking rotational symmetry.

In the spherically symmetric case, we can instead analytically continue from $n < 1$; a similar strategy has been employed for calculating entanglement in string theory [37, 36, 82, 116, 117, 132]. For $n < 1$, we can replace a neighborhood of the conical singularity with a spherical cap consisting of the region $\theta < \theta_c$ of the sphere with radius r_c:

$$ds^2 = r_c^2(d\theta^2 + \sin(\theta)^2 d\phi^2). \tag{4.33}$$

Matching the length and extrinsic curvature determines

$$r_c = \frac{\epsilon n}{\sqrt{1 - n^2}}, \qquad \theta_c = \cos^{-1}(n). \tag{4.34}$$

[5]In the CFT limit this issue does not arise, because we take $\mu \to 0$ prior to taking $\epsilon \to 0$.

The equation for the stress tensor is given by (4.32) just as in the hyperbolic case, except that for $n < 1$ it always has real solutions.

We can now use this solution to determine the constant in (4.30). Stress tensor conservation implies that u should be continuous, which fixes the singular part of u:

$$u^2 = 1 + \frac{c\mu}{24\pi r^2} + \frac{c\mu}{24\pi r^2 \sin(\theta)^2}\left(\frac{1}{n^2} - 1\right). \tag{4.35}$$

This equation determines the stress tensor

$$\begin{aligned} T_\theta^\theta &= \frac{2}{\mu}\left(1 - \sqrt{1 + \frac{c\mu}{24\pi r^2} + \frac{c\mu}{24\pi r^2}\left(\frac{1}{n^2} - 1\right)\frac{1}{\sin(\theta)^2}}\right), \\ T_\phi^\phi &= \frac{2}{\mu}\left(1 - \frac{1 + \frac{c\mu}{24\pi r^2}}{\sqrt{1 + \frac{c\mu}{24\pi r^2} + \frac{c\mu}{24\pi r^2}\left(\frac{1}{n^2} - 1\right)\frac{1}{\sin(\theta)^2}}}\right). \end{aligned}$$

The sign of u in (4.35) has been chosen so that the CFT limit $\mu \to 0$ is finite. In this limit, the stress tensor agrees with the result obtained from the Schwarzian transformation law of the stress tensor [87].

Computing the Conical Entropy Having found an appropriate boundary condition for the stress tensor, we can now proceed to calculate the entropy $\tilde{S}_n$ for $n < 1$. To find the partition function at fixed n, we vary the radius to obtain:

$$\frac{d\log Z}{dr} = -2\pi n r \int d\theta \sin(\theta)\left(T_\theta^\theta + T_\phi^\phi\right) \tag{4.36}$$

$$= \frac{4\pi n}{\mu}\int d\theta\ \sin(\theta)\left[\frac{2r^2 + \frac{c\mu}{24\pi} + \alpha^2}{\sqrt{r^2 + \alpha^2}} - r\right] \tag{4.37}$$

where we have defined

$$\alpha^2 = \frac{c\mu}{24\pi}\left(1 + \left(\frac{1}{n^2} - 1\right)\frac{1}{\sin(\theta)^2}\right). \tag{4.38}$$

This can be integrated in r to obtain

$$\log Z = \frac{n}{4G\ell}\int d\theta \sin(\theta)\left[\ell^2 \sinh^{-1}\left(\frac{r}{\alpha}\right) + r\sqrt{\alpha^2 + r^2} - r^2\right]. \tag{4.39}$$

For $n = 1$, this reduces to (4.18).

From (4.39) we obtain the entropy

$$\tilde{S}_n = -n^2 \frac{d}{dn}\left(\frac{\log Z}{n}\right) \tag{4.40}$$

$$= \frac{rc}{6n}\int d\theta \frac{1}{\sin(\theta)}\left(\frac{1-\frac{c\mu}{24\pi\alpha^2}}{\sqrt{\alpha^2+r^2}}\right) \tag{4.41}$$

$$= \frac{c}{3}\frac{(1-n^2)}{\sqrt{\frac{c\mu}{24\pi r^2}+n^2}}\Pi\left(n^2\middle|\frac{r^2+\frac{c\mu}{24\pi}}{r^2+\frac{c\mu}{24\pi n^2}}\right). \tag{4.42}$$

Π is the complete elliptic integral of the third kind:

$$\Pi(\eta|m) := \int_0^{\pi/2} \frac{d\theta}{(1-\eta\sin(\theta)^2)\sqrt{1-m\sin(\theta)^2}}. \tag{4.43}$$

This function has a branch cut for $n \geq 1$, corresponding to a pole in the integral (4.43). However, we can take the principal value of this integral, which is real for all $n > 0$. This function is displayed in figure 4.2.

The limit $n \to 0$ gives the logarithm of the rank of the reduced density matrix:

$$\tilde{S}_0 = \sqrt{\frac{2\pi c}{3\mu}}\pi r \tag{4.44}$$

which scales with the length, πr, of the boundary. This is suggestive of a lattice theory in which an interval of length L has a Hilbert space of dimension $\exp\left(\sqrt{\frac{2\pi c}{3\mu}}L\right)$.

4.2.2 Comparison with holography

We now compare our result (4.42) with the prediction from holography. According to the proposal of Ref. [44], $\tilde{S}_n$ is given by $L/4G$, where L is the length of a cosmic string whose tension induces an angle $\frac{2\pi}{n}$ in the bulk. By rescaling ϕ, this is equivalent to finding the length of a geodesic in a smooth geometry with induced boundary metric given by (4.20). Finding this solution is equivalent to finding an embedding of the metric (4.20) into Euclidean AdS_3.

This embedding problem is most easily solved in the coordinates:

$$ds^2 = \ell^2(d\varphi^2 + e^{-2\varphi}(d\rho^2 + \rho^2 d\phi^2)). \tag{4.45}$$

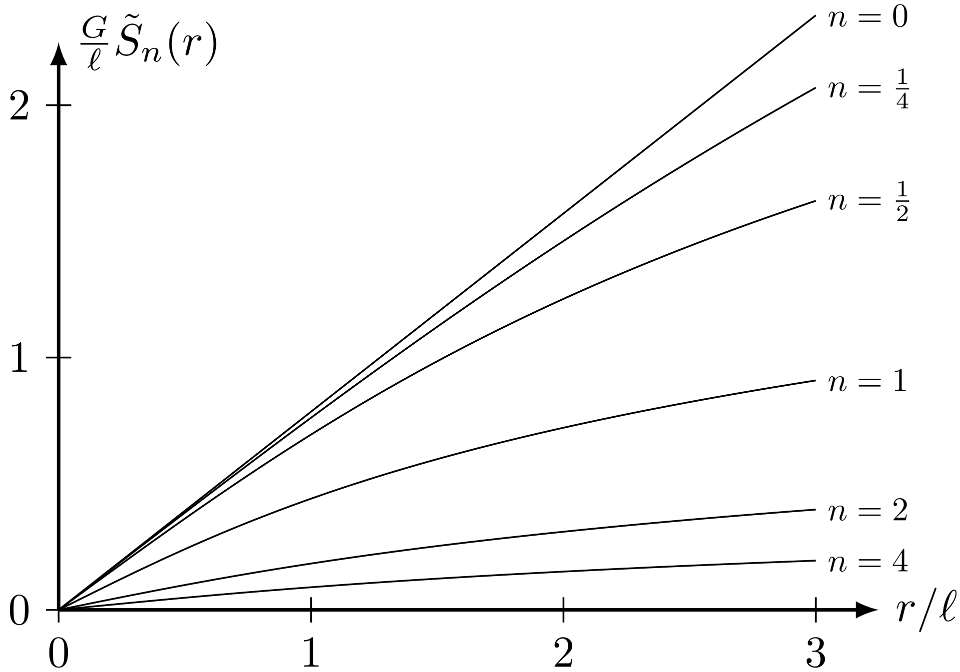

Figure 4.1: At fixed n, the entropy $\tilde{S}_n$ is a smoothly increasing function of r.

We will assume the embedding preserves rotational symmetry, so that the ϕ coordinate on the boundary is the same as the ϕ coordinate of the bulk. The embedding is then defined by two functions $\varphi(\theta)$ and $\rho(\theta)$. Demanding that the embedding be isometric leads to the equations

$$\rho = e^{\varphi}\frac{r}{\ell}n\sin(\theta), \qquad \frac{r^2}{\ell^2} = \left(\frac{d\varphi}{d\theta}\right)^2 + e^{-2\varphi}\left(\frac{d\rho}{d\theta}\right)^2. \tag{4.46}$$

This can be rewritten as a single differential equation for φ,

$$\frac{r^2}{\ell^2} = \left(\frac{d\varphi}{d\theta}\right)^2 + \frac{n^2r^2}{\ell^2}\left[\left(\frac{d\varphi}{d\theta}\right)\sin(\theta) + \cos(\theta)\right]^2, \tag{4.47}$$

which can be solved algebraically for $\frac{d\varphi}{d\theta}$. The resulting embedding in global coordinates is shown in figure 4.3.

The length of the geodesic connecting the point $\theta = 0$ and $\theta = \pi$ in the metric (4.45) is given by

$$L = \ell(\varphi(\pi) - \varphi(0)) = \ell\int_0^\pi d\theta\,\frac{d\varphi}{d\theta}. \tag{4.48}$$

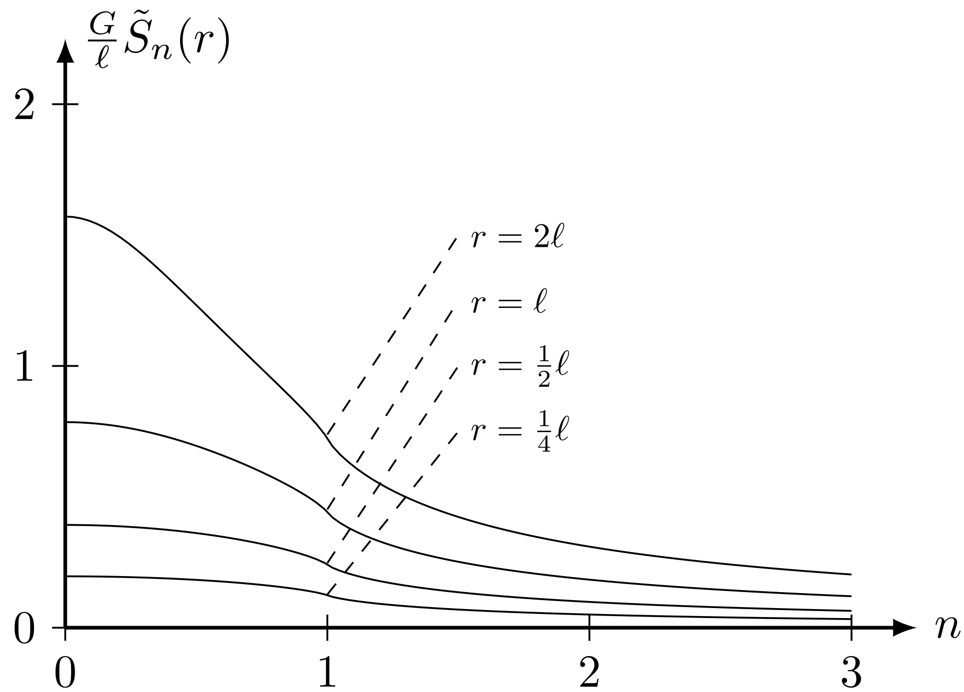

Figure 4.2: At fixed r, $\tilde{S}_n$ is a decreasing function of n, with a kink at $n = 1$.

To find the total entropy, we integrate $\frac{d\varphi}{d\theta}$

$$\frac{L}{4G} = \frac{r}{2G}\frac{1}{\sqrt{1+r^2n^2}}\left[\frac{1+r^2}{r^2}K(m) - \frac{1}{r^2}\Pi(\eta|m)\right], \tag{4.49}$$

where $K(m) = \Pi(0|m)$ is the complete elliptic integral of the first kind and

$$m = \frac{n^2(1+r^2)}{1+n^2r^2}, \qquad \eta = \frac{r^2n^2}{1+r^2n^2}. \tag{4.50}$$

Using an identity of elliptic integrals[6], this agrees with our expression (4.42) for the entropy, with the identification of μ and c given in (**??**).

[6] $(m-\eta)\Pi(\eta|m) + (m-\eta')\Pi(\eta'|m) = mK(m)$ where $(1-\eta)(1-\eta') = 1-m$ [42, Eq. 19.7.9].

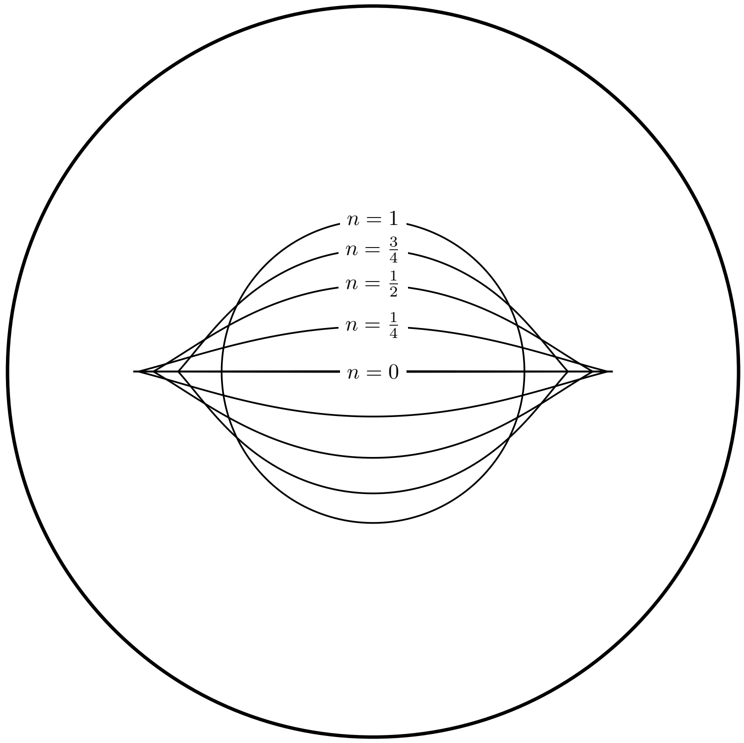

Figure 4.3: Embedded surfaces with $r = \ell = 1$ in Euclidean AdS_3. For this graph, we have used Poincaré disk coordinates in which the metric is $ds^2 = \frac{4d\vec{x}^2}{(1-\vec{x}^2)^2}$ with the third coordinate suppressed. The AdS boundary is the outer circle. At $n = 1$, the embedding is a circle. As n is decreased the embeddings have an increasingly elongated "football" shape. At $n = 0$, the embedding degenerates to a line.

4.3 Finite c von Neumann Entropy

At finite c, we take the operator $T\bar{T}$ to be defined as the coincidence limit of the bilocal operator

$$T\bar{T}(x) = 8\, G_{\mu\nu\rho\sigma}(x) \lim_{y\to x} \left(T^{\mu\nu}(x)\, T^{\rho\sigma}(y)\right). \tag{4.51}$$

This is in keeping with the definition of the T^2 operators in higher dimensions

Assume that the only deforming operator is $\mathcal{O} = T\bar{T}$, with coupling constant μ. Then, there is only one dimensionful scale in the theory: it is defined by μ, which has dimensions of length squared. This means that the expectation value of this operator can be obtained by taking a derivative with respect to μ:

$$\partial_\mu \log Z = \frac{1}{4}\int \langle T\bar{T}(x)\rangle. \tag{4.52}$$

We will promote the deformation parameter μ to a function $\mu\lambda(x)$ and ask what happens when we compute the trace of the stress tensor. In other words, that the equation (4.52) can be upgraded to

$$\frac{\delta \log Z}{\delta\lambda(x)} = \frac{\mu}{4}\langle T\bar{T}(x)\rangle. \tag{4.53}$$

Recalling that the trace of the stress tensor encodes the response of the partition function under the change of every length scale present in the theory:

$$g_{\alpha\beta}\frac{\delta \log Z}{\delta g_{\alpha\beta}} = \langle T^\mu_\mu(x)\rangle = -\mu\frac{\delta \log Z}{\delta\lambda(x)} - \frac{c}{24\pi}R(x), \tag{4.54}$$

we obtain the flow equation:

$$\langle T^\mu{}_\mu(x)\rangle = -\frac{\mu}{4}\langle T\bar{T}(x)\rangle - \frac{c}{24\pi}R(x). \tag{4.55}$$

We assume that no other scale is generated, and hence that we can extrapolate (4.55) to finite μ.

Recall that the partition function, viewed as a functional of the metric, is a generating functional for stress-tensor correlation functions:

$$\langle T^{\mu\nu}(x_1)\cdots T^{\rho\sigma}(x_n)\rangle = \frac{1}{Z[g]}\left(\frac{-2}{\sqrt{g(x_1)}}\frac{\delta}{\delta g_{\mu\nu}(x_1)}\right)\cdots\left(\frac{-2}{\sqrt{g(x_n)}}\frac{\delta}{\delta g_{\rho\sigma}(x_n)}\right)Z[g]. \tag{4.56}$$

Substituting (4.56) into the flow equation (4.55) yields a functional differential equation for the partition function $Z[\gamma]$:

$$-\frac{2}{\sqrt{\gamma(x)}}g_{\mu\nu}(x)\frac{\delta Z[g]}{\delta g_{\mu\nu}(x)} = -\lim_{y\to x}\frac{\mu\, G_{\mu\nu\rho\sigma}(x)}{\sqrt{g(x)}\sqrt{g(y)}}\,\frac{\delta^2 Z[g]}{\delta g_{\mu\nu}(x)\delta g_{\rho\sigma}(y)} - \frac{c}{24\pi}R(x)\,Z[g]. \tag{4.57}$$

This is a linear second-order functional differential equation for $Z[g]$, a functional of the two-dimensional metric g.

4.3.1 Finite c Flow equation and the Wheeler-DeWitt equation

At generic values of c, the bulk equation that the $T\bar{T}$ flow equation gets mapped to is the Wheeler de Witt equation.

It will help to introduce a change of variables to eliminate the linear term in (4.57). This is given by

$$\Psi[g] = e^{\frac{2}{\mu}\int \mathrm{d}^2x\sqrt{g}} Z[g]. \tag{4.58}$$

(4.57) then becomes the following equation for $\Psi[g]$:

$$\frac{16\pi G}{\sqrt{g}(x)\sqrt{g}(y)} G_{\mu\nu\rho\sigma}(x) \lim_{x\to y} : \frac{\delta^2 \Psi[g]}{\delta g_{\mu\nu}(x)\delta g_{\rho\sigma}(y)} : + \frac{1}{16\pi G}\left(R + \frac{2}{\ell^2}\right)\Psi[g] = 0. \tag{4.59}$$

This equation is the radial Wheeler-DeWitt equation in a space with negative cosmological constant provided we identify bulk constants as in (**??**). Eq. (4.59) can be obtained from the classical constraint equation (3.31) by replacing the momenta with functional derivatives with respect to $\gamma_{ij}(x)$. Conversely, the classical constraint equation (3.31) can be obtained from the Wheeler-DeWitt equation (4.59) in the leading-order WKB approximation, as we will see in detail in the following section.

The symbol $:(\cdots):$ denotes a procedure for regularizing the functional derivative at coincident points, which we will leave unspecified as it is unnecessary for the purposes of this work. This is because we will not attempt to solve the above second-order functional differential equation. Instead, we will specialize to the case where the constant radius slices are spheres and consider the quantization of the Hamiltonian constraint (3.31) after phase space reduction.

4.3.2 Phase space reduction for S^2 radial slices

From the point of view of quantizing the bulk theory, it is natural to introduce a gauge fixing for the Hamiltonian constraint. We choose the constant mean curvature gauge

$$\frac{\pi^\mu_\mu}{\sqrt{g}} = \tau, \tag{4.60}$$

where τ is constant. We then decompose the momentum conjugate to the metric as follows:

$$\pi^{\mu\nu} = \left(\tfrac{1}{2}\tau\sqrt{g}g^{\mu\nu} + \sigma^{\mu\nu}\right), \tag{4.61}$$

where $\sigma^{\mu\nu}$ is traceless. Imposing the constancy of τ implies additionally that

$$\nabla_\mu \sigma^{\mu\nu} = 0, \tag{4.62}$$

and therefore $\sigma^{\mu\nu}$ is transverse and tracefree. On the sphere, such tensors must vanish identically, so $\sigma^{\mu\nu} = 0$. We can then do a conformal decomposition of the metric

$$g_{\mu\nu} = e^{2\lambda(x)} h_{\mu\nu}, \tag{4.63}$$

where $h_{\mu\nu}$ is the standard round metric on the unit 2-sphere, with $R[h] = 2$. The Ricci scalar of g is $R[g] = e^{-2\lambda}(R[h] - 2\nabla^2\lambda)$. Having fixed the gauge as in (4.63) the Hamiltonian constraint for sphere radial slices becomes an equation for the conformal factor λ:

$$\Delta\lambda = \frac{R[h]}{2} - e^{2\lambda}\left(\frac{(16\pi G)^2\tau^2}{4} - \frac{1}{\ell^2}\right). \tag{4.64}$$

This equation has a unique solution up to a zero mode, which was shown in [118, 63] (although for a different combination of signs). All two-dimensional spherical geometries are conformal to the round sphere and solutions to (4.64) give us the factor with which to perform the Weyl transformation between the given metric and the round two-sphere metric. Only the global part of the Hamiltonian constraint remains unfixed. This is obtained by integrating the above equation over the sphere

$$V\left((16\pi G)^2\tau^2 - \frac{2}{\ell^2}\right) - 4\pi = 0, \tag{4.65}$$

where $V = \int d^2x\, \sqrt{h}\, e^{\lambda(x)}$ is the volume.

York's method involves taking the mean curvature τ to be 'time' and treating the volume as a true Hamiltonian

$$V(\tau) = \frac{4\pi}{(16\pi G)^2\tau^2 - \frac{2}{\ell^2}}. \tag{4.66}$$

This is the well-known deparameterization of the Hamiltonian constraint in terms of York time. However, it is not convenient for our purposes, since we are interested in the wave-function of a spherical geometry as a function of radius. In this representation, the volume is a function of the configuration variable r, $V = 4\pi r^2$. The mean curvature τ is the conjugate momentum to the volume, which is related to the momentum conjugate to r as

$$p_V = \frac{p_r}{8\pi r}. \tag{4.67}$$

With these variables, the classical constraint equation (4.66) becomes

$$G^2 p_r^2 - \left(1 + \frac{r^2}{\ell^2}\right) = 0. \tag{4.68}$$

The quantization of the above constraint equation is therefore a time-independent Schrödinger equation, and the phase space reduction necessitates limiting our attention to the global modes of the geometry. We choose to parameterize these modes through the radius r and its conjugate momentum p_r.

4.3.3 Symmetry reduced action and Wheeler-DeWitt equation

In this section, we give an alternative derivation of the Wheeler-DeWitt equation (4.77) by symmetry reduction of the action for AdS_3 gravity, c.f. [27]. This will be useful in order to make contact with the Euclidean path integral formalism in section 4.4.

The action for general relativity with a negative cosmological constant including the counterterm [14] takes the form $S_{GR} = S_{\mathrm{EH}} + S_{\mathrm{GHY}} + S_{\mathrm{CT}}$ where the various terms are given by:

$$S_{\mathrm{EH}} = -\frac{1}{16\pi G}\int \mathrm{d}^3x\sqrt{\gamma}\left(R + \frac{2}{\ell^2}\right), \tag{4.69}$$

$$S_{\mathrm{GHY}} = \frac{1}{8\pi G}\oint \mathrm{d}^2x\sqrt{g}\,K \tag{4.70}$$

$$S_{\mathrm{CT}} = \frac{1}{8\pi G\ell}\oint \mathrm{d}^2x\sqrt{g}. \tag{4.71}$$

Note that the counterterm is precisely the factor appearing in (4.58) that relates the deformed CFT partition function and the bulk gravity wavefunction:

$$\Psi[g] = e^{S_{\mathrm{CT}}} Z[g]. \tag{4.72}$$

The context, however, is slightly different from Ref. [14]. There, the counterterm was required to obtain a finite partition function in the limit where γ is large. Here it appears as a generating function in a canonical transformation that eliminates the first derivative term from the flow equation (4.57).

We wish then to pass to the Hamiltonian formalism, which first requires foliating the spacetime by hypersurfaces. We will assume spherical symmetry, under which a general metric takes the form

$$\mathrm{d}s^2 = N^2(\rho)\,\mathrm{d}\rho^2 + r^2(\rho)\,\mathrm{d}\Omega_2. \tag{4.73}$$

This form is quite familiar from studies of homogeneous and isotropic cosmology: the function $r(\rho)$ controls the size of the sphere of fixed ρ and is analogous to the scale factor in the Friedman-Robertson-Walker metric. The function $N(\rho)$ is analogous to the lapse, the difference being that in the metric (4.73) the normal to the surfaces of constant ρ is spacelike.

The action, not including the counterterm then becomes:

$$S_{\mathrm{EH}} + S_{\mathrm{GHY}} = -\frac{1}{2G} \int \mathrm{d}\rho \; N(\rho) \left(1 + \left(\frac{r'(\rho)}{N(\rho)}\right)^2 + \frac{r(\rho)^2}{\ell^2}\right). \tag{4.74}$$

We note that the Euclidean action is both negative and unbounded below. The fact that the action is negative is an important feature — we will see that it is precisely this feature that leads to the positivity of the entropy, and consistency with the Ryu-Takayanagi formula when evaluated on the classical solution. The fact that the fluctuations around the classical solution also have negative Euclidean action is a serious problem since the integral of e^{-S} will diverge. This is the famous conformal mode problem, which we will be forced to revisit in section 4.4.

The derivation of the Wheeler-DeWitt equation from the symmetry reduced action is standard. We first identify the conjugate momenta to r and N:

$$p_r = -\frac{r'}{GN}, \quad p_N = 0. \tag{4.75}$$

The latter is a constraint, and its preservation leads directly to the Hamiltonian constraint

$$G^2 p_r^2 - \left(1 + \frac{r^2}{\ell^2}\right) = 0. \tag{4.76}$$

This agrees with the result (4.68) derived from gauge-fixing in the hamiltonian formalism.

The classical limit in the bulk theory is one where $G \ll \ell$, which in terms of the field theory implies that $c \gg 1$. In that limit, the $T\overline{T}$ flow equation becomes the radial Hamiltonian constraint in the bulk. In the minisuperspace approximation, i.e. when we truncate to the symmetry reduced sector, this constraint reduces to (4.68). When we quantize this theory, we obtain an equation valid when G and ℓ are comparable, which translates into c remaining $\mathcal{O}(1)$. This is why the arbitrary c flow equation in the symmetry reduced $T\bar{T}$ theory should be identified with the minisuperspace Hamiltonian constraint.

4.3.4 Emergent diffeomorphism invariance

The phase space reduction presented in 4.3.2 shows us how the large c flow equation is written purely in terms of variations with respect to the global geometrical modes. From the intrinsically two dimensional point of view, this is to be expected given that the flow equation is written in terms of only one point functions of the stress tensor which are themselves given purely in terms of the derivatives of the partition function with respect to the radius.

However, the fact that the RG flow equation even at finite c involves only derivatives with respect to the global modes of the metric is nontrivial. This happens for the $T\bar{T}$ flow equation (in [50], [28]) on $\mathbb{T}^2$ as well, except for very different reasons [7]. There, the localization of the flow equation on to the zero-mode sector is due to the separation independence of the contracted two-point function of stress tensors whose coincidence limit defines the $T\bar{T}$ operator [99].

In our case, however, it is crucially important that the flow equation that arises from deforming a conformal field theory with $T\bar{T}$ can be rewritten as the Wheeler-DeWitt equation. The Wheeler-DeWitt equation and the Ward identity $\nabla_i \langle T^{ij} \rangle = 0$ encode the invariance of the wave function $\Psi[g]$ under normal and tangential deformations of the hypersurface on which it is evaluated. In quantum gravity, these deformations describe the action of diffeomorphisms of the bulk space-time into which the hypersurface is embedded.

One could ask how important it was to deform a holographic CFT in order to exploit bulk diffeomorphism invariance. We argue that in fact, it isn't important since all we have done is to make a change of variables and identified constants in a certain manner. When we are considering pure gravity in the bulk, this agrees with a finite cutoff generalization of the conventional AdS/CFT dictionary. However, when other matter fields are involved in the bulk, in order to maintain the identifications as dictated by the AdS/CFT dictionary, this mapping is inadequate [100] and other double trace deformations involving operators other than the stress tensor must be included in the boundary theory [80].

Given the expectation that correlation functions of local operators are expected to be smeared or delocalized by the $T\bar{T}$ deformation [29], even if the standard dictionary is maintained unless other double-trace deformations are included, the bulk theory likely also involves nonlocal matter fields.

[7]The $T\bar{T}$ flow equation is the one that reads $\partial_\mu Z = \langle T\bar{T}(x) \rangle Z$, and it tells us how the quantum field theory responds under the change of one scale in the problem, i.e. the one associated to the $T\bar{T}$ operator. The RG flow equation or Callan–Symanzik equation on the other hand tells us how the theory responds to a local change of scale

However, one can also consider the stress tensor sector of a general CFT at finite c, which isn't expected to possess a classical bulk dual. In such a theory, we can apply the $T\bar{T}$ deformation and find that the local RG flow equation will take the form (4.57). After making some simple identification of the constants and by redefining the partition function in terms of $\Psi[g]$, this flow equation takes the form of the Wheeler-DeWitt equation (4.59), irrespective of whether or not it has a semiclassical bulk dual. [8]

In this chapter, we are interested in computing the sphere partition function in a $T\bar{T}$ deformed CFT with some arbitrary, finite central charge. Although the method involves exploiting the emergent bulk diffeomorphism invariance in order to turn our problem into an effectively quantum mechanical one, we do not require the theory we are deforming to possess a holographically dual description in terms of string theory on AdS_3.

4.4 Sphere partition function

The quantization of the reduced phase space Hamiltonian constraint leads to the following spherically symmetric Wheeler-DeWitt equation:

$$G^2 \left(\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{2b-1}{r} \frac{\mathrm{d}}{\mathrm{d}r} \right) \psi(r) = \left(1 + \frac{r^2}{\ell^2} \right) \psi(r), \tag{4.77}$$

where $\psi(r) = \Psi[\gamma_{S^2}]$ is the wavefunctional evaluated on a sphere of radius r. The constant b appears due to the ordering ambiguity for the kinetic term. Then, if we recall the relationship between the partition function and the solution to the Wheeler-DeWitt equation (4.58), the sphere partition function of the $T\bar{T}$ deformed CFT is given by

$$Z(r) = e^{-\frac{r^2}{2G\ell}} \psi(r). \tag{4.78}$$

It will be convenient to work in terms of gravitational units and set $\ell = 1$. In these units, G becomes a dimensionless parameter, the ratio of the Planck length to the AdS radius, which is small in the classical limit. Although the equation admits an exact solution in terms of special functions, it will be instructive to first study the solution semiclassically.

[8] We acknowledge that Aitor Lewkowycz independently realized this perspective on $T\bar{T}$ deformed theories

4.4.1 WKB approximation

When G is small, the equation (4.77) can be treated by the WKB approximation. It will be convenient to define $\psi = e^W$, where $W = \log Z + S_{\rm CT}$ is the effective action, up to the counterterm. We then expand in powers of G,

$$W = \frac{1}{G}W_0 + W_1 + GW_2 + \dots . \tag{4.79}$$

In terms of the effective action, equation (4.77) becomes

$$G^2\left(W'' + (W')^2 + \frac{2b-1}{r}W'\right) = (1+r^2). \tag{4.80}$$

where $'$ denotes $\frac{d}{dr}$.

For a second-order equation, there are two classical solutions $W_0^{\pm}$. We can then consider the expansion around each of these solutions, and a general solution is given by:

$$\psi(r) = \alpha_+ e^{\frac{1}{G}W_0^+ + W_1^+ + \dots} + \alpha_- e^{\frac{1}{G}W_0^- + W_1^- + \dots} \tag{4.81}$$

The negative solution W_0^- is suppressed relative to the positive solution by the exponential of the classical action. This is non perturbatively small when r is larger than the Planck scale. For now, we focus on corrections around the dominant saddle point.

Classical solution At leading order in the G expansion we have two solutions for W_0,

$$W_0'(r) = \pm\sqrt{r^2+1}. \tag{4.82}$$

The positive solution is given by

$$W_0^+(r) = \frac{1}{2}(\sinh^{-1}(r) + r\sqrt{r^2+1}). \tag{4.83}$$

This solution corresponds to the Euclidean action (without the holographic counterterm) evaluated on the classical saddle point, which in this case is a region of Euclidean AdS space bounded by a sphere of radius r. Restoring ℓ and the counterterm, this corresponds to a classical solution

$$Z(r) \sim \exp\left(\frac{\ell}{2G}\left(\sinh^{-1}\left(\frac{r}{\ell}\right) + \frac{r}{\ell}\sqrt{\frac{r^2}{\ell^2}+1} - \frac{r^2}{\ell^2}\right)\right) \tag{4.84}$$

This agrees with the evaluation of the classical action for a region of Euclidean AdS_3 bounded by a sphere of radius r [49].

The other classical solution $W_0^-(r)$ corresponds to the opposite sign of (4.83). This yields a saddle point whose contribution to the partition function is exponentially suppressed when $r \gg G$.

One-loop correction The WKB expansion also allows us to find subleading corrections in the loop expansion. Substituting the leading-order WKB solution into the first order equation yields

$$W_1'(r) = -\frac{1}{2}\left(\frac{2b-1}{r} + \frac{W_0''(r)}{W_0'(r)}\right). \tag{4.85}$$

The one-loop correction is given by

$$W_1(r) = -\tfrac{1}{2}(2b-1)\log(r) - \tfrac{1}{4}\log(1+r^2). \tag{4.86}$$

Note that the one-loop correction is the same around both classical solutions $W_0^{\pm}$, since (4.85) is unaffected by flipping the sign of W_0.

Note also that the corrections become large as $r \to 0$, while the leading term $W_0(r)$ vanishes in that limit. This indicates that the WKB approximation breaks down at distances approaching the Planck scale.

From this result, we can infer the leading order quantum corrected partition function in the bulk:

$$Z(r) \sim \exp\left(\frac{\ell}{2G}\left(\sinh^{-1}\left(\frac{r}{\ell}\right) + \frac{r}{\ell}\sqrt{\frac{r^2}{\ell^2}+1} - \frac{r^2}{\ell^2}\right) + \frac{1}{4}\log\left(\frac{r^2}{r^2+\ell^2}\right) - b\log\left(\frac{r}{\ell}\right)\right). \tag{4.87}$$

There are two important caveats with this solution: we have neglected the contribution of the subdominant saddle point, and we have not fixed the constant in front of the solution. To appropriately resolve this issue we require boundary conditions as $r \to 0$. Since the point $r = 0$ is outside the regime of validity of the WKB approximation, we have to use other methods to determine the solution in that regime.

4.4.2 Exact solution

Equation (4.77) admits an exact solution. Changing to the independent variable to $z = r^2/G$ and rescaling ψ to g as

$$g(z) = e^{z/2}\psi(\sqrt{Gz}), \tag{4.88}$$

(4.77) becomes Kummer's equation ([42], §13)

$$zg''(z) + (b - z)g'(z) - ag(z) = 0, \tag{4.89}$$

where $a = \frac{1}{4G} + \frac{b}{2}$. When $b \notin \mathbb{Z}$, the general solution of (4.89) is given by a linear combination of the confluent hypergeometric functions $U(a, b, z)$ and $M(a, b, z)$.[9] For $b \in \mathbb{Z}$ the expansion for $z \to 0$ is more complicated and so we will assume the generic case $b \notin \mathbb{Z}$ from here on.

In the limit $z \to 0$, the solution is parameterized by

$$g(z) = c_1 M(a, b, z) + c_2 z^{1-b} M(a - b + 1, 2 - b, z). \tag{4.90}$$

As $z \to 0$,

$$g(z) = c_1(1 + O(z)) + c_2 z^{1-b}(1 + O(z)). \tag{4.91}$$

The constants c_1, c_2 are determined from the boundary conditions as $z \to 0$. In the classical solution (4.83), we only need a single boundary condition since the leading order WKB equation is first-order. However, the Wheeler-DeWitt equation is second-order and so we require a second boundary condition to fully specify the solution. This additional boundary condition determines the contribution from the subdominant saddle point.

The boundary condition chosen in Ref.[49] was to take $Z = 1 + O(r^2)$ as $r \to 0$. The simplest choice which achieves this is $c_1 = 1$, $c_2 = 0$. This leads to a partition function that coincides with that of a trivial theory in the ultraviolet, and we will see in section 4.5 that this makes the entanglement entropy vanish as $r \to 0$ as might be expected if the $T\overline{T}$ deformation acts as an effective ultraviolet cutoff.[10]

4.4.3 Path integral representation

The preceding results can also be obtained from a Euclidean path integral. This will be useful in the following section in comparing the entanglement entropy with the bulk length.

We first deparametrize the system by introducing a parameter L be the diameter of the bulk, i.e. twice the proper distance from the center to a sphere of fixed radius r. The wavefunction $\psi(r)$ is replaced with a wavefunction $\psi(r, L)$ in which L plays the rôle

[9] $M(a, b, z)$ is sometimes denoted ${}_1F_1(a; b; z)$.

[10] An alternative prescription [71] is to start from the conformal field theory in the limit $r \to \infty$. In this limit, the partition function has an undetermined constant coming from the cutoff scale.

of a Euclidean time coordinate. The Wheeler-DeWitt equation is then replaced with the Euclidean Schrödinger equation:

$$\left(-4G\partial_L - G^2\left(\partial_r^2 + \frac{2b-1}{r}\partial_r\right) + r^2 + 1\right)\psi(r, L) = 0. \tag{4.92}$$

The solution of the Wheeler-DeWitt equation will be obtained by integration,

$$\psi(r) = \int \mu_L dL\, \psi(r, L). \tag{4.93}$$

The measure factor μ_L is required by dimensional analysis and has units of inverse length. We also leave the contour of integration unspecified; provided $\psi(r, L)$ solves (4.92), then the integral (4.93) will solve the Wheeler-DeWitt equation if the contour is chosen such that the contribution from the endpoints vanishes.

We can solve this with an ansatz $\psi(r, L) = e^{\alpha(L)+\beta(L)r^2}$, which leads to a pair of equations

$$4G\frac{\mathrm{d}}{\mathrm{d}L}\beta + 4G^2\beta^2 - 1 = 0 \tag{4.94}$$

$$4G\frac{\mathrm{d}}{\mathrm{d}L}\alpha + 4bG^2\beta - 1 = 0. \tag{4.95}$$

These equations are easily integrated, but two constants of integration must be specified. One constant can be absorbed into a redefinition of L (and hence in a shift of the contour used in integrating L). The second constant can be absorbed into the measure factor μ_L. Having made these choices, the solution is given by:

$$\beta = \frac{1}{2G}\coth\left(\frac{L}{2}\right) \tag{4.96}$$

$$\alpha = \frac{L}{4G} - b\log\sinh\left(\frac{L}{2}\right). \tag{4.97}$$

Which yields

$$\psi(r, L) = \sinh\left(\frac{L}{2}\right)^{-b}\exp\left[\frac{L}{4G} + \frac{r^2}{2G}\coth\left(\frac{L}{2}\right)\right]. \tag{4.98}$$

Before carrying out the path integral, we first look at the classical solutions. The exponential has two real saddle points $\pm L_0$, where

$$\sinh(L_0/2) = r. \tag{4.99}$$

The positive saddle point at $L = L_0$ corresponds to Euclidean AdS_3 with line element

$$\mathrm{d}s^2 = \mathrm{d}\rho^2 + \sinh(\rho)^2 \mathrm{d}\Omega_2 \tag{4.100}$$

which has $r = \sinh(\rho)$ and $L = 2\rho$. Evaluating $\psi(r, L)$ at this point yields

$$\psi(r, L_0) = r^{-b} \exp\left[\frac{1}{2G}\left(\sinh^{-1}(r) + r\sqrt{r^2+1}\right)\right] \tag{4.101}$$

The quantity in the exponential is precisely the classical solution W_0^+ given by the WKB method (4.83). The saddle point at $L = -L_0$ gives the exponentially suppressed WKB solution W_0^-. [11]

We now turn to carry out the Euclidean path integral, and the choice of contour for L.

The conformal mode problem: Since L is a Euclidean length, it would seem natural to integrate over positive real L. However, the result (4.98) diverges as e^L for large L, and as $e^{1/L}$ for small L. This is a manifestation of the **conformal mode problem**.

The Euclidean Einstein–Hilbert action is not bounded from below on the space of real metrics. In perturbation theory, when the action is expanded around a flat background, we find that the scalar mode corresponding to the fluctuations of the conformal mode comes with the wrong sign in its kinetic term. In particular, if we can find a gauge where the metric can be decomposed as:

$$g_{ab} = e^{\phi} \bar{g}_{ab} \tag{4.102}$$

where $\bar{g}_{ab}$ is a metric with fixed determinant, then the offending mode in the perturbative analysis is the fluctuation field φ:

$$\phi = \phi_o + \varphi, \tag{4.103}$$

where ϕ_o is the background value of the conformal factor, which on flat space is zero.

In minisuperspace, the only dynamical field is the zero mode of ϕ, in other words, the scale factor. Therefore, we expect to face the full brunt of the conformal mode problem when we study the quantum theory. And indeed, the action isn't bounded from below, even at the fully nonlinear level. This in turn prevents us from carrying out straight forward path integral quantization of this system.

[11]There are an infinite number of complex saddle points at $L = \pm L_0 + 2\pi i n$ for $n \in \mathbb{Z}$. Shifting the imaginary part of $L \to L + 2\pi i n$ shifts the integrand as $\psi(r, L + 2\pi i n) = \psi(r, L) e^{\frac{\pi i n}{2G}}$, so the complex contours and saddle points differ from their real counterparts by a phase.

One way to remedy this problem would be to analytically continue to complex metrics, which on minisuperspace just means that we should analytically continue to complex values of the scale factor. Then, to perform the path integral over the complexified scale factor, a contour of integration has to be chosen. It must be such that at the very least, the partition function converges.

There is not however a single agreed-upon prescription for carrying out gravitational path integrals of this type; but a list of desired criteria were outlined in Ref. [78].

There are other remedies for this issue in the literature in the context of performing the full Euclidean path integral (i.e. beyond symmetry reduction in $d > 3$). One such prescription is to perform a Wick rotation in a 'proper time gauge' of the metric fluctuations, as discussed in [38]. Another way to circumvent the issue involves a nonlocal field redefinition [113]. In both of these contexts, a Jacobian arising from the path integral measure cancels the divergence of the Euclidean action.

We know that on general grounds, whatever contour we choose to integrate (4.93), it should be deformable to a combination of steepest descent contours which will pass through some set of saddle points. In order to match with the expected classical behavior, this set of saddle points must include the positive saddle L_0. We will also demand that the solution $\psi(r)$ is real: the original Euclidean integral we want to deform, though divergent, is formally real, and we will see in section 4.5 that complex solutions for $\psi(r)$ lead to complex entropy.

The steepest descent curves passing through the real saddle points have a stationary phase, which means the quantity in the exponential of (4.98) is real. Letting $L = x + iy$, these curves are solutions of

$$y + 2r^2 \frac{\sin(y)}{\cos(y) - \cosh(x)} = 0 \tag{4.104}$$

The solutions, displayed in figure 4.4, consist of the real line, together with a loop encircling the origin. Starting from the positive saddle L_0, the steepest descent contour leaves the real axis along the loop and intersects the negative saddle $-L_0$. Starting from $-L_0$ the steepest descent contour covers the negative real axis.

To carry out the integral defined by (4.93) we introduce a substitution $w = \coth(L/2)$. Under this substitution we obtain

$$\psi(r) = -2\mu_L \int dw (w+1)^{\frac{1}{4G}+\frac{b}{2}-1}(w-1)^{-\frac{1}{4G}+\frac{b}{2}-1} e^{\frac{r^2}{2G}w}. \tag{4.105}$$

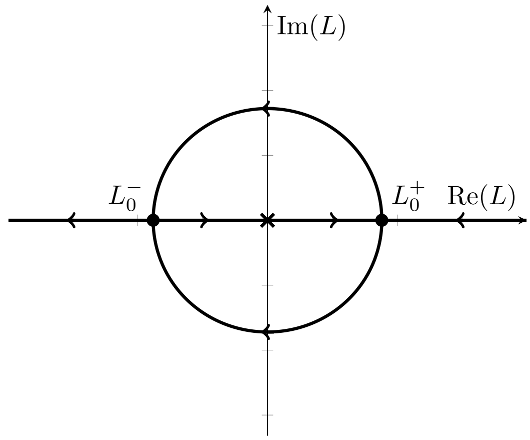

Figure 4.4: Steepest descent contours for the evaluation of the integral of $\psi(r, L)$. The thick lines denote values for which $\psi(r, L)$ is real; here we have set $r = 1$, but the qualitative behavior is independent of r. The direction of the arrows denotes the direction in which the integrand is decreasing. The cross at the origin denotes an essential singularity of the integrand.

The saddle points at $\pm L_0$ are mapped to the points $\pm w_0$ where $w_0 = \sqrt{1 + 1/r^2}$. The positive L-axis is mapped to $w > 1$, while the negative L axis is mapped to $w < -1$. These two lines are separated by a branch cut for $-1 < w < 1$; crossing the branch cut shifts the imaginary part of L.

We can carry out the integral along either steepest descent contour using known integral representations of the confluent hypergeometric functions ([42], §13.4)

$$I_1 := \int_{-1}^{-\infty} dw\ (w+1)^{a-1}(w-1)^{-a+b-1} e^{\frac{z}{2}w} = -\Gamma(a)\, 2^{b-1} e^{-z/2} U(a, b, z), \tag{4.106}$$

$$I_2 := \oint_\gamma dw\ (w+1)^{a-1}(w-1)^{-a+b-1} e^{\frac{z}{2}w} = 2\pi i \frac{\Gamma(a)}{\Gamma(1+a-b)} 2^{b-1} e^{-z/2} \mathbf{M}(a, b, z). \tag{4.107}$$

We have reintroduced the variables a, z from subsection 4.4.2. The curve γ encircles the interval $(-1, 1)$ clockwise. We see that these integrals give solutions to the Wheeler-DeWitt equation, as required. I_1 is real, while I_2 is imaginary.

The simplest solution to the conformal mode problem is simply to rotate the contour to the negative real axis, resulting in a convergent integral [66]. In [17], it was shown how conformal bootstrap can be used to resolve the ambiguities associated with the analytic continuation of the integration over the Weyl mode in two space-time dimensions. However, we see here that the contour does not pass through the saddle point at L_0, so gives a partition function that is exponentially suppressed. The resulting entropy is negative: it is given by $-L_0/4G$ in the classical limit. Thus the naive resolution to the conformal mode problem gives an unphysical result in our application.

Instead, we can consider a contour passing through the saddle point at L_0. This contour terminates on another saddle point, the one at $-L_0$. In cases such as this when the steepest descent curve intersects another saddle point, the saddle point is said to be on a Stokes line. We must decide how to extend the contour past the other saddle point. The standard method to deal with this case is to analytically continue the parameters of the problem to complex numbers: for example, by giving G a small imaginary part, $G \to G \pm i\epsilon$. When we do this, the steepest descent contour passing through L_0 slightly misses the saddle point at $-L_0$ and continues close to the negative real axis. As we take $\epsilon \to 0$ the contour becomes a union of the loop encircling the origin and the negative real axis.

However, depending on the sign of the imaginary part of G, the contour will traverse $(-\infty, 0)$ in either the positive or negative direction. Thus the real part of $\psi(r)$ will be discontinuous as a function of the complexified G. A natural prescription, in this case, is to take the average of the two results [4]. This cancels out the contribution from the negative real axis, and the result is proportional to the loop integral (4.107). While this gives a purely imaginary integral, the result for $\psi(r)$ can be made real by choosing an imaginary value for the measure factor μ_L.

We can further choose the measure factor μ_L so that the partition function satisfies $Z(r) \to 0$ as $r \to 0$. The result is

$$\psi(r) = e^{-r^2/2G} M\left(\frac{1}{4G} + \frac{b}{2}, b, \frac{r^2}{2G}\right). \tag{4.108}$$

This is the same as the exact solution obtained in section 4.4.2.

We note that other choices of the contour are possible. which could also include contributions from complex saddle points. Each of these contributions comes with a nontrivial phase, but they can be summed over to give a real result. We note, for example, that by choosing the Pochhammer contour in carrying out the integral (4.105), we obtain a result proportional to $M(a, b, z)$ c.f. ([42], 13.4.11).

4.5 Entanglement Entropy

We now consider entanglement entropy in the $T\overline{T}$-deformed theory. Specifically, we will consider the setup of Ref. [49], where the conformal field theory lives on a sphere and the entangling surface consists of two antipodal points. This corresponds to the entanglement entropy of the Hartle-Hawking state of the CFT, viewed as a state on a circle divided into two semicircles. Equivalently, it is the de Sitter entropy of the CFT.

This calculation was carried out in Ref. [49] in the strict large c limit. This corresponds to the classical limit of the bulk gravity theory, and in this limit, the result reproduces the Ryu-Takayanagi formula [137].

However, our result also includes $1/c$ corrections, which correspond to quantum corrections in the bulk. These corrections to the entanglement entropy are conjectured to capture entanglement of the bulk fields across the minimal surface [57]:

$$S = \frac{\langle L \rangle}{4G} + S_{\text{bulk}} + O(1/c). \tag{4.109}$$

In the present case, the bulk theory is pure gravity and has no local degrees of freedom, so it is not clear exactly what bulk degrees of freedom could be responsible for S_{bulk}.

Higher-order corrections in the $1/c$ expansion generically are expected to deform the location of the extremal surface to a *quantum extremal surface* [54]. In our case, the location of the bulk surface is fixed by rotational symmetry, so the quantum extremal surface coincides with the extremal surface at all orders: a geodesic through the center of the bulk. In this case, the higher-order corrections to the entanglement entropy of the boundary are higher-order corrections in the semiclassical expansion of $\frac{\langle L \rangle}{4G} + S_{\text{bulk}}$ about the classical minimal surface.

Even when a theory has no local degrees of freedom, it still has an entanglement entropy. The best-known example is Chern-Simons theory, where the entanglement comes from edge modes localized on the entangling surface [151, 59, 156]. Since 3D gravity is closely related to Chern-Simons theory [153], we might expect that the bulk entanglement entropy has a similar description in terms of edge modes.

To precisely describe entanglement entropy in terms of bulk gravity, we need a description of the gravitational edge modes and their multiplicities. It is not known how to do this for general relativity, but some aspects of the problem at the classical level were worked out in Ref. [49]. In 3D gravity there are a number of more specific proposals , see e.g. [115, 65, 152]. We expect the area to play a preferred role in the edge modes for gravity, based on the algebra and generators and also an analogy with the Ryu-Takayanagi formula [79, 106]. In 3D gravity, the total length is the only invariant of the intrinsic geometry of the entangling surface. Moreover, calculations in holography [45, 2] show that the entanglement spectrum is flat for fixed area states, at least to leading order in the $1/N$

expansion. This suggests that the gravitational edge modes in 3D are labeled by the length of the entangling surface, with a multiplicity given by $\exp\left(\frac{L}{4G}\right)$ in the classical limit.

We will find some evidence for this picture, namely that the corrections to the entropy of the boundary theory capture fluctuations of the length of the bulk geodesic. However, this interpretation relies on choosing a real contour for the gravitational path integral. The interpretation of the entropy as fluctuations of the bulk length operator is obscured when the length is continued to complex values. We will comment on this further in the discussion.

4.5.1 Entropy for antipodal points on the sphere

We first briefly review the calculation of the entanglement entropy in the special case of antipodal points, and its relation to the sphere partition function. We are essentially repeating the argument of Ref. [49].

The entropy is computed through the replica trick, which first involves evaluating the partition function on an n-sheeted branched cover of the sphere, where the branch points are at the entangling surface. The line element on such a space is:

$$\mathrm{d}s^2 = r^2[\mathrm{d}\theta^2 + n^2\sin^2(\theta)\mathrm{d}\phi^2]. \tag{4.110}$$

The entanglement entropy is then given by the formula:

$$S = (1 - n\partial_n)\log Z|_{n=1}. \tag{4.111}$$

In the absence of rotational symmetry, the partition function must be analytically continued to a neighbourhood of $n = 1$. In the present situation, rotational symmetry allows us to vary n continuously.

Under infinitesimal variations of n, the generating functional responds as follows:

$$-\int \mathrm{d}^2x\sqrt{\gamma}\langle T^{\phi}{}_{\phi}\rangle = \frac{\mathrm{d}\log Z}{\mathrm{d}n}|_{n=1}. \tag{4.112}$$

In the limit $n \to 1$, the full spherical symmetry is re-instated, so the one point function of the energy momentum tensor is isotropic $\langle T^{\mu\nu}\rangle = \alpha(r)g^{\mu\nu}$. This in particular tells us that $\langle T^{\phi}{}_{\phi}\rangle = \frac{1}{2}\langle T^{\mu}{}_{\mu}\rangle$. This means that the von Neumann entropy in the situation at hand can be computed through the formula:

$$S = \left(1 - \frac{r}{2}\frac{\mathrm{d}}{\mathrm{d}r}\right)\log Z. \tag{4.113}$$

Here Z is the sphere partition function without a conical singularity.

Thus we can obtain the entanglement entropy directly from the sphere partition function. The formula (4.113) is the usual thermodynamic formula for the entropy in terms of the partition function but with r^2 playing the role of inverse temperature β. We will return to this point shortly.

Note that the counterterm introduces a shift in the partition function of the form $\log Z \to \log Z + \alpha r^2$ where α is constant. This shift of the partition function does not change the entanglement entropy (4.113), so the counterterm drops out of the calculation of the entropy.

4.5.2 Loop expansion of the entropy

It is straightforward to apply this formula to the partition function in the WKB approximation, yielding

$$S = \frac{1}{2G}\sinh^{-1}\left(\frac{r}{\ell}\right) - \frac{1}{2}(2b-1)\log(r) - \frac{1}{4}\log(1+r^2) + \frac{2b-1}{4} + \frac{1}{4}\frac{r}{1+r^2}. \qquad (4.114)$$

The first term, propotional to $1/G$, is the classical term $\frac{L_0}{4G}$. The remaining terms are a one-loop quantum correction. We recall that this solution is ambiguous up to the addition of a constant, which can be fixed by the choice of boundary conditions.

Another limit of interest is one where $r \gg \ell$. In the large r limit, we find

$$S = \left(\frac{\ell}{2G} - b\right)\log r + O(r^0) = \left(\frac{c}{3} - b\right)\log(r) + O(r^0). \qquad (4.115)$$

This gives a correction to the CFT result, which is small if we hold b fixed in the large c limit.

4.5.3 Finite radius FLM corrections

We recall that the partition function takes the form

$$Z(r) = \int \mu_L dL \sinh\left(\frac{L}{2}\right)^{-b} \exp\left[\frac{L}{4G} + \frac{r^2}{2G}\coth\left(\frac{L}{2}\right)\right], \qquad (4.116)$$

up to a counterterm which does not affect the entropy. Provisionally, we will treat (4.116) as though it were a convergent real integral, returning to the issues of the conformal mode in due course.

We note that (4.116) resembles a canonical thermal partition function in which the states are labelled by L, with density of states dn and energy E,

$$Z = \int dn(L) e^{-\beta E(L)}. \tag{4.117}$$

In this equation we identify r^2 with the inverse temperature β; this is consistent with the formula (4.113) for the entropy. The density of states and energy can be read off from (4.116) as:

$$dn(L) = \mu_L dL \, \sinh\left(\frac{L}{2}\right)^{-b} \exp\left[\frac{L}{4G}\right], \tag{4.118}$$

$$E(L) = -\frac{1}{2G} \coth\left(\frac{L}{2}\right). \tag{4.119}$$

Including the counterterm simply shifts $E(L)$ by a constant.

We can now relate the entropy calculated by the sphere trick to fluctuations of the length L. The distribution over lengths implied by this canonical distribution is given by the measure

$$d\rho(L) = \frac{1}{Z} e^{-\beta E(L)} dn(L). \tag{4.120}$$

Since L is a continuous parameter, the entropy of the distribution $d\rho$ is not invariant under reparametrizations of L. Instead, one should consider the relative entropy $S(\rho\|\sigma)$ where σ is a reference distribution:

$$S(\rho\|\sigma) = \int d\rho(L) \log\left(\frac{d\rho(L)}{d\sigma(L)}\right). \tag{4.121}$$

This quantity is invariant under reparametrizations when both $d\rho$ and $d\sigma$ transform as measures. Note that the sign is opposite from the one appearing in the entropy, $S = -\sum p \log p$.

Having put the partition function into the canonical form, we can straightforwardly calculate the entropy. It takes the suggestive form

$$S = \frac{\langle L \rangle}{4G} - S(\rho\|\sigma), \tag{4.122}$$

where $\langle L \rangle$ denotes the expectation value in the distribution ρ. The reference distribution σ is defined by

$$d\sigma(L) = \mu_L dL \sinh\left(\frac{L}{2}\right)^{-b}. \tag{4.123}$$

This suggests an interpretation in which the gravitational edge modes are labelled by the length L of the bulk geodesic. The number of distinct eigenvalues of L is given by the measure $d\sigma(L)$, while the degeneracy of the eigenvalues is given by $e^{L/4G}$. This would appear to give a realization of the Faulkner-Lewkowycz-Maldacena proposal [57] in which the bulk entanglement entropy can be understood as arising from gravitational edge modes labeled by the length.

Unfortunately, such a nice interpretation seems to be precluded by the conformal mode problem. When the contour for the L integral is complex, the interpretation of the states being labeled by a real geometric length is not available. We do not know whether there is any interpretation of the entropy analogous to (4.122) when L is allowed to be complex.

Chapter 5

$T\bar{T}$ partition functions as solutions to the Wheeler de Witt equation

In this section, we will see how the partition function of a $T\bar{T}$ deformed quantum field theory can directly be obtained by a functional integral transformation of the undeformed theory's partition function. The integral will be over the frame fields of the geometry on which the theory lives, weighted by a Gaussian kernel. As such, this definition circumvents the need to solve a flow equation in order to obtain the partition function, although we will indeed see that the flow equation does follow from this alternative definition. When considering the deformation of conformal field theories, we will in fact see that the Callan–Symanzik equation that the partition function solves can be rewritten as the Wheeler de Witt equation in three dimensions.

5.1 The Integral Kernel and the Flow equation

The integral transformation referred to in the paragraph above is given by

$$Z[f] = \int \mathcal{D}e \, \exp\left[-\frac{1}{\mu}\int \epsilon^{\alpha\beta}\epsilon_{ab}(e-f)^a_\alpha(e-f)^b_\beta\right] Z_0[e]\,. \tag{5.1}$$

Here, e^a_μ denotes the zweibein on the space where the field theory lives, and f^a_μ denotes the one on the space on which the deformed theory lives. In the appendix [], basic relationships between the objects in the first order and metric variables will be summarised. $Z_0[e]$ denotes the undeformed theory's partition function.

For notational convenience, we introduce:

$$K_\mu(e,f) = -\frac{1}{\mu}\int \epsilon^{\alpha\beta}\epsilon_{ab}(e-f)^a_\alpha(e-f)^b_\beta \tag{5.2}$$

To see that this defines the partition function of the $T\bar{T}$ deformed theory, we take the derivative with respect to μ:

$$\partial_\mu Z[f] = \int d^2x\left(\frac{1}{2}\varepsilon^{ab}\varepsilon_{\mu\nu} : \frac{\delta}{\delta f^a_\mu(x)}\frac{\delta}{\delta f^b_\nu(x)} :\right) Z[f], \tag{5.3}$$

where the 'normal ordering' is defined not by a coincident limit of any sort but simply as

$$\frac{1}{2}\varepsilon^{ab}\varepsilon_{\mu\nu} : \delta_{f^a_\mu(x)}\delta_{f^a_\nu(x)} := \frac{1}{2}\varepsilon^{ab}\varepsilon_{\mu\nu}\delta_{f^a_\mu(x)}\delta_{f^a_\nu(x)} + \frac{2}{\lambda}\delta^{(2)}(0). \tag{5.4}$$

This is all the normal ordering one needs to do to get the flow equation (5.3) to work; if it turns out that there are more divergences on the RHS, they also drive this flow.

Before proving the equation, let us define the one- and two-point functions of the stress tensor. The one-point function is defined by

$$\langle T^\mu_a(x)\rangle = -(\det f(x))^{-1}\frac{1}{Z[f]}\frac{\delta}{\delta f^a_\mu(x)}Z[f], \tag{5.5}$$

which means that

$$\delta(-\log Z) = \int (\det f)\delta f^a_\mu \langle T^\mu_a\rangle. \tag{5.6}$$

Similarly, the two-point function is defined as

$$\langle T^\mu_a(x)T^\nu_b(y)\rangle = \frac{1}{Z[f]}\frac{1}{\det f(x)\det f(y)}\delta_{f^a_\mu(x)}\delta_{f^b_\nu(y)}Z[f], \tag{5.7}$$

where the $\det f$ factors are outside so that the change in the free energy is a double integral of the two-point function. With this definition, the RHS of (5.3) (up to the normal ordering) is [1]

$$\int d^2x(\det f(x))\ \varepsilon^{ab}\epsilon_{\mu\nu}\frac{1}{(\det f(x))^2}\delta_{f^a_\mu}\delta_{f^b_\nu}Z = Z\int d^2x \det f \varepsilon^{ab}\epsilon_{\mu\nu}\langle T^\mu_a T^\nu_b\rangle \tag{5.8}$$

[1]Recall that ϵ is a tensor density. ε on the other hand is simply the Levi-Civita symbol. Hence, $\epsilon_{\mu\nu} = \det(f)\varepsilon_{\mu\nu}$.

Thus, it appears a sensible generalisation of the $T\bar{T}$ operator.

Moving on to prove the kernel satisfies the flow equation, we see the left hand side of (5.3) becomes simply

$$\partial_\mu Z[f] = \int \mathcal{D}e \left(\frac{1}{2\mu^2} \int \varepsilon_{ab} (f-a)^a \wedge (f-e)^b \right) e^{-K_\mu} Z_0[e]. \tag{5.9}$$

Let us first work out the right hand side, without the normal-ordering, keeping any new contact terms which may arise:

$$\begin{aligned} \frac{1}{2} \int d^2x \ \varepsilon^{ab} \varepsilon_{\mu\nu} \frac{\delta}{\delta f^a_\mu(x)} \frac{\delta}{\delta f^b_\nu(x)} Z[f] &= \int d^2x \ \varepsilon^{ab} \varepsilon_{\mu\nu} \frac{\delta}{\delta f^a_\mu(x)} \int \mathcal{D}e \ \left(-\frac{1}{\mu} \varepsilon_{bc} \varepsilon^{\nu\rho} (f-e)^c_\rho(x) \right) e^{-K_\mu} Z_0[e] \\ &= -\frac{2}{\mu}\delta(0) \left(\int d^2x \right) Z + \int \mathcal{D}e \ \left(\frac{1}{2\mu^2} \int \varepsilon_{ab} (f-a)^a \wedge (f-e)^b \right) e^{-K_\mu} Z_0[e]. \end{aligned} \tag{5.10}$$

We thus define the normal-ordering by subtracting out the piece proportional to $\delta(0)$. Note this term exists equally well in flat space. With this prescription, the flow equation(5.3) holds rather trivially:

$$\partial_\mu \log Z[f] = \left\langle \frac{1}{\mu^2} \int (f-e)^a \wedge (f-e)^b \right\rangle \tag{5.11}$$

$$= \frac{1}{Z[f]} \int d^2x \varepsilon^{ab} \varepsilon_{\mu\nu} \frac{1}{2} : \frac{\delta}{\delta f^a_\mu} \frac{\delta}{\delta f^b_\nu} : Z[f]. \tag{5.12}$$

In fact, the weight in the exponent $K_\mu(e,f)$ is itself the action of ghost-free massive gravity in two dimensions. This connection was highlighted in [].

The equation (5.1) was introduced in [61], but towards a different end. There, it was shown that when this integral transformation is applied to a conformal field theory partition function satisfies:

$$\left\{ f^a_\mu(x) \delta_{f^a_\mu(x)} - \frac{c}{24\pi} (\det f) R[f](x) \right\} Z_0[f] = 0, \tag{5.13}$$

then the deformed partition function satisfies:

$$\left\{ f^a_\mu(x) \frac{\delta}{\delta f^a_\mu(x)} + \mu \varepsilon^{ab} \varepsilon_{\mu\nu} : \frac{\delta}{\delta f^a_\mu(x)} \frac{\delta}{\delta f^b_\nu(x)} : - \frac{c-24}{24\pi} \det(f) R[f](x) \right\} Z[f] = 0. \tag{5.14}$$

This is the three dimensional Wheeler de Witt equation.

This definition justifies the assumption made in the previous chapter where the local Callan–Symanzik equation of the deformed theory is identified with three dimensional the Wheeler de Witt equation.

5.2 Annular Path Integral

The wavefunction $Z[f]$, being a functional of the vielbein f on the 2d slice, is expressed in the analog of the position basis [2]

$$Z[f] = \langle f|\Psi\rangle \tag{5.15}$$

Satisfying $\hat{C}_\alpha|\Psi\rangle = 0$, the state $|\Psi\rangle$ resides in the physical Hilbert space. It is also convenient to write it in the analog of the momentum basis

$$|\Psi\rangle = \int \mathcal{D}\pi\ \Psi[\pi]|\pi\rangle = \int \mathcal{D}\pi \left(\int \mathcal{D}e\, e^{\int_\Sigma \frac{\mu}{2}\varepsilon^{ab}\pi_a\wedge\pi_b - \pi_a\wedge e^a} Z_0[e] \right) |\pi\rangle \tag{5.16}$$

where we used the overlap $\langle\pi|e\rangle = e^{-\int_\Sigma \pi_a\wedge e^a}$.

We stress the state $|\beta\rangle = \int \mathcal{D}eZ_0[e]|e\rangle$, built purely from the CFT partition function, does not solve the second-order Wheeler-de Witt equation (but rather "only" the first-order conformal anomaly equation). This makes clear that even an infinitesimal deformation radically alters the nature of the state.

An annular path integral corresponds to a transition amplitude. The two sets of boundary conditions on either side encode the initial and final state data. In 3d gravity, transition amplitudes involving at least one physical state reduce to an overlap. Indeed, since the total Hamiltonian is simply the sum of constraints C_α, which annihilate any physical state, we have

$$\langle\phi|e^{-s\hat{H}_{tot}}|\Psi_{\text{phys}}\rangle = \langle\phi|e^{-s\sum_\alpha \hat{C}_\alpha}|\Psi_{\text{phys}}\rangle = \langle\phi|\Psi_{\text{phys}}\rangle \tag{5.17}$$

We may thus equally view the partition function $Z[f]$ as a transition amplitude between the state f, corresponding to fixing the vielbein on one of the 2d boundaries, and the state $|\Psi_\lambda\rangle$ on the other.

Let us spell out the connection to the 3d gravity path integral and the choice of boundary conditions a bit further. We will show how a semi-classical treatment of our asymptotic boundary state connects to the boundary conditions discussed in [76]. Inserting a resolution of the identity, we can write

[2]Dirichlet boundary conditions in quantum gravity are notoriously tricky. We will mostly gloss over those subtleties in this discussion.

$$
\begin{aligned}
Z[f] &= \int \mathcal{D}e \langle f| e^{-s\hat{H}_{tot}} |e\rangle \Psi_\lambda[e] \\
&= \int \mathcal{D}e \langle f| e^{-s\hat{H}_{tot}} |e\rangle \left(\int \mathcal{D}\pi e^{-\int \pi_a \wedge e^a + \frac{\lambda}{2}\varepsilon^{ab}\pi_a \wedge \pi_b - \log Z_0[e]} \right)
\end{aligned} \tag{5.18}
$$

The 3d gravity path integral of interest is schematically then:

$$
Z[f] = \int \mathcal{D}e \Psi[e] \int_{E|_{r_c}=f}^{E|_\infty = e} \mathcal{D}E \, e^{-S_{GR}(E)} \tag{5.19}
$$

Here $S_{GR}(E)$ is the action for three-dimensional gravity in first-order variables with a negative cosmological constant along with the appropriate boundary terms needed for finiteness and for a well-posed variational principle. The integration variable E is the bulk vielbein and we gauge fix to the $n = 0$ gauge. The boundary values are related to the vielbeins e and f that feature in the kernel.

In the classical limit, we may evaluate $Z_\lambda[f]$ via a steepest descent approximation:

$$
Z[f] \to e^{-\int \bar{\pi}_a \wedge \bar{e}^a - \frac{\mu}{2}\varepsilon^{ab}\bar{\pi}_a \wedge \bar{\pi}_b - S_0[\bar{e}]} \tag{5.20}
$$

where $\bar{\pi}$ and $\bar{e}$ satisfy the saddle-point equations

$$
\bar{e}^a_\mu - \mu \varepsilon^{ab} \bar{\pi}_{b,\mu} = 0 \quad \bar{\pi}_{a,\mu} + \varepsilon_{\mu\nu} \det(\bar{e}) \langle T^\nu_a \rangle_0 = 0 \tag{5.21}
$$

where, in this limit,

$$
\langle T^\nu_a \rangle_0 = -\det(\bar{e})^{-1} \frac{\delta S_0[e]}{\delta e^a_\mu}\Big|_{e=\bar{e}} \tag{5.22}
$$

becomes the on-shell stress tensor of the undeformed theory.

As for the asymptotic boundary conditions, we echo the insight of [76]. They argued that $T\bar{T}$ should be treated like any other double trace deformation in holography which leads to a change of boundary conditions at infinity (see [**?**]). In particular, the fixed dyad boundary conditions at infinity in the undeformed setting should turn into mixed boundary conditions that involve both the dyad and its radial derivative or its conjugate momentum when the deformation is turned on.

How this works is that at large c (or N in dimensions greater than two) the action of the deformed theory reads

$$
S_{def} = S_o + \mu \int \left(T\bar{T} \right) . \tag{5.23}
$$

We then take a variation of this action to find

$$\delta S_{def} = \int T^{\mu}_{(o)a} \delta e^{a}_{\mu} + \mu \delta \int \left(T\bar{T} \right) = \int T^{\mu}_{(\mu)a} \delta \tilde{e}^{a}_{\mu}, \tag{5.24}$$

where, $T^{\mu}_{(o)a}$ is the stress tensor of the seed theory which couples to its source, the dyad e^a. Similarly, $T^{\mu}_{(\mu)a}$ is the deformed theory's stress tensor, which couples to a new source $\tilde{e}^a$. As was shown in [76], the latter is given by

$$\tilde{e}^{a}_{\mu} = e^{a}_{\mu} - \mu \epsilon^{ab} \epsilon_{\mu\nu} T^{\nu}_{(\mu)b}. \tag{5.25}$$

Note that by having to hold $\tilde{e}^a$ fixed, the above variation vanishes. In the bulk, this is equivalent to the statement that the bulk action that S_{def} is on shell and, through the holographic dictionary, also a function of solely the boundary data. In particular, the variation of the action on shell with an appropriate boundary term corresponding to the $T\bar{T}$ deformation added is given by the symplectic potential:

$$\delta S_{o.s.} \propto \int \pi^{\mu}_{a} \wedge \delta \tilde{e}^{a}_{\mu}, \tag{5.26}$$

where the RHS is integrated over the boundary, and π^{μ}_{a} is the momentum conjugate to the dyad $\tilde{e}^{a}_{\mu}$ induced on the boundary. The canonical transformation needed to get from the phase space parameterized by (e^a, π_a) to the $(\tilde{e}^a, \pi_a)$ is given by:

$$\tilde{e}^{a} = e^{a} - \frac{\delta W(\pi)}{\delta \pi_a}, \tag{5.27}$$

where

$$W(\pi) = 2\mu \int \epsilon^{ab} \pi_a \wedge \pi_b. \tag{5.28}$$

This is indeed the boundary term in the three dimensional gravity theory that corresponds to the $T\bar{T}$ deformation.

The specification of fixed $\tilde{e}^a$ boundary conditions, therefore, corresponds to finding some subspace of phase space on which the symplectic form computed from this potential vanishes, i.e. to a Lagrangian submanifold.

Translating the condition (5.25) into bulk language, we see that the mixed boundary condition that the $T\bar{T}$ deformation leads to is one where

$$\tilde{e}^{a}_{\mu} = e^{a}_{\mu} - \mu \epsilon^{ab} \epsilon_{\mu\nu} \pi^{\nu}_{b}, \tag{5.29}$$

is fixed at the boundary. Again, we see that the radial momentum is playing the role of the stress tensor. The function F in the path integral (5.19) is therefore $\tilde{e}$ written as a function of radial derivatives of the dyad instead of the momenta.

As mentioned before, since the state $|\Psi\rangle$ satisfies the Wheeler de Witt equation, it can be computed from a radial slice arbitrarily close to the $r = r_c$ surface. Thus, schematically, we compute the path integral between these two surfaces as:

$$\int_{E|_{r_o}=f}^{E^{(o)}|_\infty=(e-\epsilon*\pi)} \mathcal{D}E\, \exp\left(-S_{3d}(E)\right) = \int_{bc} \mathcal{D}\pi\mathcal{D}e$$
$$\exp\left(\int_{\Sigma_{r=r_c}} F_{r_o} - \int_{\Sigma_{r=\infty}} F_\infty\right) \exp\left(\int_{AdS_3} \pi_a \wedge \dot{e}^a\right). \tag{5.30}$$

Here bc stands for the boundary conditions at the $r = r_c$ surface and the surface at infinity brought to its vicinity. The functions F_{r_c} and F_∞ are the boundary terms at the $r = r_c$ surface and infinity respectively. In first-order variables, the first term is zero, and the second term is a combination of the CFT generating functional, and the term $W(\pi^{(\infty)})$ that generates the canonical transformation corresponding to the $T\bar{T}$ deformation discussed above. The vanishing of the Hamiltonian means that the phase space action involves only the kinetic term $\int \pi_a \wedge \dot{e}^a$.

Then, noting that the two surfaces are arbitrarily close to each other, we can decompose the path integral over the fields $(e_a(r,x), \pi^a(r,x))$ into $((e^{(r_c)a}(x), e^{(\infty)a}(x)), (\pi_a^{(r_c)}(x), \pi_a^{(\infty)}(x)))$:

$$= \int \mathcal{D}\pi^{(r_c)}\mathcal{D}e^{(r_c)}\mathcal{D}\pi^{(\infty)}\mathcal{D}e^{(\infty)} \exp\left(-\int \pi_a^{(r_c)} \wedge \left(e^{(r_c)} - f\right)^a\right)\times$$
$$\exp\left(-\int \pi_a^{(\infty)} \wedge \left(e^{(\infty)} - e^{(r_c)}\right)^a\right) \exp\left(-\frac{\mu}{2}\int \epsilon^{ab}\pi_a^{(\infty)} \wedge \pi_b^{(\infty)} - W^{CFT}[e^{(\infty)}]\right). \tag{5.31}$$

The integral over the fields at $r = r_c$ can be performed straightforwardly to obtain

$$\int \mathcal{D}\pi\mathcal{D}e\, \exp\left(-\int \pi_a(e-f)^a - \frac{\mu}{2}\int \epsilon^{ab}\pi_a \wedge \pi_b\right) Z_{CFT}[e] =$$
$$\int \mathcal{D}e\, e^{-\frac{1}{2\mu}\int \varepsilon_{ab}(f-e)^a\wedge(f-e)^b} Z_0[e] = Z[f]. \tag{5.32}$$

where we dropped the (∞) superscript for brevity.

We, therefore, recover the Freidel kernel formula for the $T\bar{T}$ deformed partition function.

5.3 Reduced kernel and the S^2 partition function

The prescription described in this section to define the $T\bar{T}$ deformed partition function can be applied to the case of the S^2 partition function. In the previous chapter, we described how this calculation can be done by solving the reduced Wheeler de Witt equation- which reduces to a Schrodinger like equation. It would, therefore, be overkill to use the general kernel to compute this quantity.

It will be more convenient to write this equation in terms of the variable $r = e^{\Omega}$ where Ω is the conformal factor:

$$-\frac{G_N^2}{2}\left(\partial_r^2 + \frac{k+1}{r}\partial_r\right) Z(r) + \frac{1}{2}\left(\frac{r^2}{\ell^2}+1\right) Z(r) = 0. \tag{5.33}$$

This is identical to the equation that was solved in [48], and also discussed earlier in [26], [27]. Introducing $z = r^2/G_N$, and re-scaling Z to g as:

$$g(z) = \frac{e^{z/2}}{2} Z(\sqrt{G_N z}), \tag{5.34}$$

we find that the reduced WdW equation (5.33) becomes Kummer's equation:

$$z\partial_z^2 g + \left(\frac{k}{2}+1-z\right)\partial_z g - a g(z) = 0, \tag{5.35}$$

where $a = \frac{1}{4G_N} + \left(\frac{k+2}{4}\right)$. The general solution to this equation reads

$$g(z) = c_1\, {}_1F_1\left(a, \frac{k}{2}+1, z\right) + c_2 z^{-\frac{k}{2}}\, {}_1F_1\left(a-\frac{k}{2}, 1-\frac{k}{2}, z\right). \tag{5.36}$$

Boundary conditions must be chosen in order to fix c_1 and c_2, but having done that we will have a one parameter family of radial wave-functions. It turns out that for a special value of k, there is in fact a way to both indirectly obtain the solution to this equation and fix the boundary conditions.

5.3.1 The reduced Kernel

For the purposes of this subsection, let us choose $k = -1$, and so the equation we want to solve is

$$-\frac{G_N^2}{2}\partial_r^2 Z(r) + \frac{1}{2}\left(\frac{r^2}{\ell^2}+1\right) Z(r) = 0. \tag{5.37}$$

Given that we want solutions to this equation with AdS asymptotics, we need that at large r, up to a counter term, $Z_\lambda(r) \ \sim Z_{CFT}(r)$ where the CFT partition function solves the Weyl anomaly condition on S^2:

$$R\partial_R Z_{CFT}(R) = \frac{c}{3} Z_{CFT}(R). \tag{5.38}$$

It turns out that we can in fact write the solution to the differential equation (5.37) as an integral transformation of the CFT partition function satisfying (5.38), which takes the form:

$$Z_\lambda(r) = \frac{e^{\frac{2\pi r^2}{\mu}}}{2} \int \mathrm{d}R \left(\frac{R}{\epsilon}\right)^b e^{-\frac{4\pi}{\mu}(R-r)^2} Z_{CFT}(R), \tag{5.39}$$

where b parameterizes our ignorance of the measure factors that might have entered due to, for example ratios of determinants that the gauge fixing porcedure leads to. For our purposes, we will just assume that it is an arbitrary real parameter. The solution to (5.38) is given by

$$Z_{CFT}(R) = \left(\frac{R}{\epsilon}\right)^{\frac{c}{3}}, \tag{5.40}$$

and so in all, we have

$$Z_\lambda(r) = \frac{1}{2} e^{\frac{r^2}{\lambda}} \int \mathrm{d}R e^{-\frac{2}{\mu}(R-r)^2} \left(\frac{R}{\epsilon}\right)^{\frac{c}{3}+b}. \tag{5.41}$$

We note that this integral can be thought of as the following Mellin transformation [3]:

$$Z(r) = \frac{e^{\frac{2\pi r^2}{\mu}}}{2} \mathcal{M}_R \left(e^{-\frac{4\pi}{\mu}(R-r)^2}, b + \frac{c}{3} + 1\right). \tag{5.42}$$

Which gives us:

$$Z(r) = 2^{-\frac{b}{2}-\frac{c}{6}-3} e^{-\frac{2\pi r^2}{\mu}} \left(\frac{\mu}{2\pi}\right)^{\frac{1}{6}(3b+c)} \left(4r\Gamma\left(\frac{1}{6}(3b+c+6)\right) {}_1F_1\left(\frac{1}{6}(3b+c+6); \frac{3}{2}; \frac{4\pi r^2}{\mu}\right) + \right.$$

$$\left. + \sqrt{2}\sqrt{\frac{\mu}{2\pi}} \Gamma\left(\frac{1}{6}(3b+c+3)\right) {}_1F_1\left(\frac{1}{6}(3b+c+3); \frac{1}{2}; \frac{4\pi r^2}{\mu}\right)\right). \tag{5.43}$$

[3] Here we take the definition of the Mellin transformation to be $\mathcal{M}_t(f(t), s) = \frac{1}{2}\int_0^\infty dt t^{s-1} f(t)$

This function solves the equation

$$-\frac{\mu}{16\pi}\partial_r^2 Z(r)+\frac{\pi r^2}{\mu}Z(r)+\frac{1}{4}\left(1+2b+\frac{2c}{3}\right)Z_\mu(r)=0,$$

which is identical to (5.37) if we make the identifications:

$$\frac{\mu}{8\pi\left(1+2b+\frac{2c}{3}\right)}=G_N^2,\ \ \frac{\mu}{8\pi}\left(1+2b+\frac{2c}{3}\right)=l^2. \tag{5.44}$$

Like in the full kernel, if we demand that at the reduced level

$$\lim_{\lambda\to 0}\left(e^{\frac{r^2}{4G_N\ell}}Z(r)\right)=Z_{CFT}(r), \tag{5.45}$$

then we find that we have to set $b=0$.

Now we can ask what values for c_1 and c_2 the above solution picks, and we find:

$$c_1=2^{-\frac{c}{6}-\frac{5}{2}}\left(\frac{\mu}{2\pi}\right)^{c/6+1/2}\Gamma\left(\frac{c+3}{6}\right),\ c_2=2^{-\frac{c}{6}-\frac{5}{2}}\left(\frac{\mu}{2\pi}\right)^{c/6}2\sqrt{2}\Gamma\left(\frac{c}{6}+1\right). \tag{5.46}$$

This connects back to the results from the previous chapter, except the choice of boundary conditions at $r=\infty$ are different here.

Chapter 6

Concluding Remarks

Let us summarise what's been done in this thesis: We started by considering the quantum renormalization group, which is a very general recipe to construct bulk duals from re-organizing the renormalization group flow of a given quantum field theory. We looked at the case where it is applied towards the construction of pure general relativity in the bulk, which led us to the question of how general covariance is encoded in QRG. This question is answered in the form of the holographic Wess–Zumino consistency condition, which posits a relationship between the anomalous Weyl Ward identity and the diffeomorphism Ward identity. This condition fixes the form of the flow equation that the theory living on the cutoff radial slice needs satisfy and therefore the bulk Hamilton-Jacobi equations as well. Then, when considering the limit as the cutoff is taken to infinity, we saw how we can recover the classic results pertaining to the holographic anomaly. In particular, in four-dimensional holographic CFTs, the a and c holographic anomaly coefficients must be equated.

Then we looked for what flows satisfy this consistency condition. There, we saw that for large c theories in $D = 2$ and for large N matrix field theories in higher dimensions, the consistency conditions picked out the $T\bar{T}$, and more generally the $T^2 + OO$ deformation of holographic CFTs. It was also shown how the superpotential relations for holographic RG flows in $D = 4$ were picked out as a consequence of covariance.

Then we looked at the computation of holographic entanglement entropy in two-dimensional holographic CFTs deformed by the $T\bar{T}$ operator. There, we saw how the holographic entropy conjectures of Ryu and Takayanagi and also that of Dong for Conical entropies held even int he finite cutoff setting.

Finally, we went beyond the regime of classical gravity in the bulk and looked at how the partition functions of $T\bar{T}$ deformed CFTs at some finite central charge furnish solutions to the three dimensional Wheeler de Witt equation.

We saw how the conformal mode problem affects the calculation of the bulk dual to the entanglement entropy at finite c. We propose a particular analytic continuation of the fields to circumvent this problem in minisuperspace, but the question is whether there is a nonperturbative way to implement such an analytic continuation. Also, what is the meaning of such a continuation from the quantum field theory's perspective? Analytic continuation in the space of couplings has been extensively applied in resummation techniques. It would be fascinating to see whether there is a Borel like resummation calculation on the field theory side which mirrors the analytic continuation we performed.

There are of course many questions pertaining to other topics in this thesis that remain to be answered. Some of them are listed below.

Although the emphasis here is on the emergence of a direction of space that plays a dual role to the RG scale of the holographic field theory, an important and deep question to ponder is whether time can emerge in such a manner. In one version of de Sitter holography, this is indeed the hypothesis: the quantum field theory now inhabits a spacelike boundary of dS space and the energy scale associated with this theory becomes the time like direction in the bulk. Although these models are fascinating, since the bulk theories that have so far been studied typically involve an infinite tower of massless fields interacting non locally at the cosmological scale (e.g. Vasiliev's higher spin gravity), it is hard to straight-forwardly probe the phenomenon of the emergence of the time direction. The opportunity that the deformations described in this thesis provide is an alternate mechanism that might instantiate a version of de Sitter holography with space-like boundaries.

As for the bulk Wheeler de Witt equation, we clearly see that the situation in dimensions greater than two is substantially more complicated away form large N. It is entirely unclear whether the T^2 deformation can be unambiguously defined, which is dual to the problem of finding a suitable regularization of the kinetic term in the WdW equation which involves coincident functional variations with respect to the metric. One route to progress may lie in following [27], where the authors find that minsuperspace calculations of the wavefunction in AdS_4 mirror the calculation of sphere partition functions in ABJM theory that are normally obtained through supersymmetric localization. This mysterious fact warrants more attention and the connection between supersymmetric localization on the boundary and minisuperspace localization in the bulk must be more thoroughly studied. Perhaps in so doing, there might be some way of 'defining' the T^2 operator in $D = 3$ for this very particular setting.

Given that the TT OPE in $D > 2$ does not close on to the energy-momentum sector, the worry might be that there is no hope of defining a proper WdW equation via holography in $D > 2$. However, this fact might just be a way forward, beyond canonical quantum gravity where new degrees of freedom are inevitable if one wants to define solutions of the WdW equation through holography. The question then is, what ought to replace the WdW equation? Perhaps it is some version of a Schrodinger equation for string fields. What kind of bulk symmetry does the Wess–Zumino consistency condition for the generator of local RG transformations in these deformed theories capture?

References

[1] Ofer Aharony, Shouvik Datta, Amit Giveon, Yunfeng Jiang, and David Kutasov. Modular invariance and uniqueness of $T\bar{T}$ deformed CFT. *JHEP*, 01:086, 2019.

[2] Chris Akers and Pratik Rath. Holographic Renyi Entropy from Quantum Error Correction. *JHEP*, 05:052, 2019.

[3] Emil T. Akhmedov. Notes on multitrace operators and holographic renormalization group. In *Workshop on Integrable Models, Strings and Quantum Gravity Chennai, India, January 15-19, 2002*, 2002.

[4] Inês Aniceto and Ricardo Schiappa. Nonperturbative Ambiguities and the Reality of Resurgent Transseries. *Commun. Math. Phys.*, 335(1):183–245, 2015.

[5] Sinya Aoki, Janos Balog, Tetsuya Onogi, and Peter Weisz. Flow equation for the large N scalar model and induced geometries. *PTEP*, 2016(8):083B04, 2016.

[6] Sinya Aoki, Janos Balog, Tetsuya Onogi, and Peter Weisz. Flow equation for the scalar model in the large N expansion and its applications. *PTEP*, 2017(4):043B01, 2017.

[7] Sinya Aoki, Janos Balog, and Shuichi Yokoyama. Holographic computation of quantum corrections to the bulk cosmological constant. *arXiv:1804.04636[hep-th]*, 2018.

[8] Sinya Aoki, Kengo Kikuchi, and Tetsuya Onogi. Generalized Gradient Flow Equation and Its Applications. *PoS*, LATTICE2015:305, 2016.

[9] Sinya Aoki and Shuichi Yokoyama. AdS geometry from CFT on a general conformally flat manifold. *Nucl. Phys.*, B933:262–274, 2018.

[10] Sinya Aoki and Shuichi Yokoyama. Flow equation, conformal symmetry, and anti-de Sitter geometry. *PTEP*, 2018(3):031B01, 2018.

[11] Richard L. Arnowitt, Stanley Deser, and Charles W. Misner. The Dynamics of general relativity. *Gen. Rel. Grav.*, 40:1997–2027, 2008.

[12] Meseret Asrat, Amit Giveon, Nissan Itzhaki, and David Kutasov. Holography Beyond AdS. *Nucl. Phys.*, B932:241–253, 2018.

[13] Joseph J. Atick and Edward Witten. The Hagedorn Transition and the Number of Degrees of Freedom of String Theory. *Nucl. Phys.*, B310:291–334, 1988.

[14] Vijay Balasubramanian and Per Kraus. A Stress tensor for Anti-de Sitter gravity. *Commun. Math. Phys.*, 208:413–428, 1999.

[15] Aritra Banerjee, Arpan Bhattacharyya, and Soumangsu Chakraborty. Entanglement Entropy for TT deformed CFT in general dimensions. *Nucl. Phys.*, B948:114775, 2019.

[16] Florent Baume, Boaz Keren-Zur, Riccardo Rattazzi, and Lorenzo Vitale. The local Callan-Symanzik equation: structure and applications. *JHEP*, 08:152, 2014.

[17] Teresa Bautista, Atish Dabholkar, and Harold Erbin. Quantum Gravity from Time-like Liouville theory. 2019.

[18] C. Becchi, S. Giusto, and C. Imbimbo. The Wilson-Polchinski renormalization group equation in the planar limit. *Nucl. Phys.*, B633:250–270, 2002.

[19] Jibril Ben Achour and Etera R. Livine. Protected $SL(2,\mathbb{R})$ Symmetry in Quantum Cosmology. *JCAP*, 1909:012, 2019.

[20] Massimo Bianchi, Daniel Z. Freedman, and Kostas Skenderis. Holographic renormalization. *Nucl. Phys.*, B631:159–194, 2002.

[21] Martin Bojowald, Suddhasattwa Brahma, Umut Buyukcam, and Fabio D'Ambrosio. Hypersurface-deformation algebroids and effective spacetime models. *Phys. Rev.*, D94(10):104032, 2016.

[22] Giulio Bonelli, Nima Doroud, and Mengqi Zhu. $T\bar{T}$-deformations in closed form. *JHEP*, 06:149, 2018.

[23] J. David Brown and M. Henneaux. Central Charges in the Canonical Realization of Asymptotic Symmetries: An Example from Three-Dimensional Gravity. *Commun. Math. Phys.*, 104:207–226, 1986.

[24] Pasquale Calabrese and John Cardy. Entanglement entropy and conformal field theory. *J. Phys.*, A42:504005, 2009.

[25] Pasquale Calabrese and Alexandre Lefevre. Entanglement spectrum in one-dimensional systems. *Phys. Rev. A*, 78:032329, Sep 2008.

[26] Pawel Caputa, Shouvik Datta, and Vasudev Shyam. Sphere partition functions and cut-off AdS. 2019.

[27] Pawel Caputa and Shinji Hirano. Airy Function and 4d Quantum Gravity. *JHEP*, 06:106, 2018.

[28] John Cardy. The $T\overline{T}$ deformation of quantum field theory as random geometry. *JHEP*, 10:186, 2018.

[29] John Cardy. $T\overline{T}$ deformation of correlation functions. 2019.

[30] S Carlip. A phase space path integral for (21)-dimensional gravity. *Classical and Quantum Gravity*, 12(9):2201–2207, sep 1995.

[31] Horacio Casini, Marina Huerta, and Robert C. Myers. Towards a derivation of holographic entanglement entropy. *JHEP*, 05:036, 2011.

[32] Andrea Cavaglià, Stefano Negro, István M. Szécsényi, and Roberto Tateo. $T\bar{T}$-deformed 2D Quantum Field Theories. *JHEP*, 10:112, 2016.

[33] Soumangsu Chakraborty, Amit Giveon, Nissan Itzhaki, and David Kutasov. Entanglement Beyond AdS. 2018.

[34] Bin Chen, Lin Chen, and Peng-Xiang Hao. Entanglement entropy in $T\overline{T}$-deformed CFT. *Phys. Rev.*, D98(8):086025, 2018.

[35] Steven Corley. A Note on holographic Ward identities. *Phys. Lett.*, B484:141–148, 2000.

[36] Atish Dabholkar. Quantum corrections to black hole entropy in string theory. *Phys. Lett.*, B347:222–229, 1995.

[37] Atish Dabholkar. Strings on a cone and black hole entropy. *Nucl. Phys.*, B439:650–664, 1995.

[38] A. Dasgupta and R. Loll. A Proper time cure for the conformal sickness in quantum gravity. *Nucl. Phys.*, B606:357–379, 2001.

[39] Shouvik Datta and Yunfeng Jiang. $T\bar{T}$ deformed partition functions. *JHEP*, 08:106, 2018.

[40] Jan de Boer, Erik P. Verlinde, and Herman L. Verlinde. On the holographic renormalization group. *JHEP*, 08:003, 2000.

[41] Avinash Dhar and Spenta R. Wadia. Noncritical strings, RG flows and holography. *Nucl. Phys.*, B590:261–272, 2000.

[42] *NIST Digital Library of Mathematical Functions.* Release 1.0.18 of 2018-03-27. F. W. J. Olver, A. B. Olde Daalhuis, D. W. Lozier, B. I. Schneider, R. F. Boisvert, C. W. Clark, B. R. Miller and B. V. Saunders, eds.

[43] Brian P. Dolan. Symplectic geometry and Hamiltonian flow of the renormalization group equation. *Int. J. Mod. Phys.*, A10:2703–2732, 1995.

[44] Xi Dong. The Gravity Dual of Renyi Entropy. *Nature Commun.*, 7:12472, 2016.

[45] Xi Dong, Daniel Harlow, and Donald Marolf. Flat entanglement spectra in fixed-area states of quantum gravity. 2018.

[46] Xi Dong, Eva Silverstein, and Gonzalo Torroba. De Sitter Holography and Entanglement Entropy. 2018.

[47] William Donnelly and Laurent Freidel. Local subsystems in gauge theory and gravity. *JHEP*, 09:102, 2016.

[48] William Donnelly, Elise LePage, Yan-Yan Li, Andre Pereira, and Vasudev Shyam. Quantum corrections to finite radius holography and holographic entanglement entropy. *arXiv:1909.11402[hep-th].*

[49] William Donnelly and Vasudev Shyam. Entanglement entropy and $T\overline{T}$ deformation. *Phys. Rev. Lett.*, 121:131602, 2018.

[50] Sergei Dubovsky, Victor Gorbenko, and Guzmán Hernández-Chifflet. $T\overline{T}$ partition function from topological gravity. *JHEP*, 09:158, 2018.

[51] Sergei Dubovsky, Victor Gorbenko, and Mehrdad Mirbabayi. Asymptotic fragility, near AdS_2 holography and $T\overline{T}$. *JHEP*, 09:136, 2017.

[52] Roberto Emparan. Black hole entropy as entanglement entropy: A Holographic derivation. *JHEP*, 06:012, 2006.

[53] Netta Engelhardt. Into the Bulk: A Covariant Approach. *Phys. Rev.*, D95(6):066005, 2017.

[54] Netta Engelhardt and Aron C. Wall. Quantum Extremal Surfaces: Holographic Entanglement Entropy beyond the Classical Regime. *JHEP*, 01:073, 2015.

[55] Johanna Erdmenger. A Field theoretical interpretation of the holographic renormalization group. *Phys. Rev.*, D64:085012, 2001.

[56] Szilard Farkas and Emil J. Martinec. Gravity from the Extension of Spatial Diffeomorphisms. *J. Math. Phys.*, 52:062501, 2011.

[57] Thomas Faulkner, Aitor Lewkowycz, and Juan Maldacena. Quantum corrections to holographic entanglement entropy. *JHEP*, 11:074, 2013.

[58] Thomas Faulkner, Hong Liu, and Mukund Rangamani. Integrating out geometry: Holographic Wilsonian RG and the membrane paradigm. *JHEP*, 08:051, 2011.

[59] Jackson R. Fliss, Xueda Wen, Onkar Parrikar, Chang-Tse Hsieh, Bo Han, Taylor L. Hughes, and Robert G. Leigh. Interface Contributions to Topological Entanglement in Abelian Chern-Simons Theory. *JHEP*, 09:056, 2017.

[60] Adrián Franco-Rubio and Guifre Vidal. Entanglement and correlations in the continuous multi-scale entanglement renormalization ansatz. *JHEP*, 12:129, 2017.

[61] Laurent Freidel. Reconstructing AdS/CFT. 2008.

[62] Daniel Friedan and Anatoly Konechny. Gradient formula for the beta-function of 2d quantum field theory. *J. Phys.*, A43:215401, 2010.

[63] Yoshihisa Fujiwara and Jiro Soda. Teichmüller Motion of (2+1)-Dimensional Gravity with the Cosmological Constant. *Progress of Theoretical Physics*, 83(4):733–748, 04 1990.

[64] Dmitri V. Fursaev and Sergey N. Solodukhin. On the description of the Riemannian geometry in the presence of conical defects. *Phys. Rev.*, D52:2133–2143, 1995.

[65] Marc Geiller. Edge modes and corner ambiguities in 3d Chern–Simons theory and gravity. *Nucl. Phys.*, B924:312–365, 2017.

[66] G. W. Gibbons, S. W. Hawking, and M. J. Perry. Path Integrals and the Indefiniteness of the Gravitational Action. *Nucl. Phys.*, B138:141–150, 1978.

[67] Gaston Giribet. $T\bar{T}$-deformations, AdS/CFT and correlation functions. *JHEP*, 02:114, 2018.

[68] Amit Giveon, Nissan Itzhaki, and David Kutasov. $\mathrm{T}\overline{\mathrm{T}}$ and LST. *JHEP*, 07:122, 2017.

[69] Amit Giveon, Nissan Itzhaki, and David Kutasov. A solvable irrelevant deformation of AdS_3/CFT_2. *JHEP*, 12:155, 2017.

[70] Henrique Gomes and Vasudev Shyam. Extending the rigidity of general relativity. *J. Math. Phys.*, 57(11):112503, 2016.

[71] Victor Gorbenko, Eva Silverstein, and Gonzalo Torroba. dS/dS and $T\overline{T}$. *JHEP*, 03:085, 2019.

[72] Eric Gourgoulhon. 3+1 formalism and bases of numerical relativity. *arXiv:gr-qc/0703035*, 2007.

[73] Sebastian Grieninger. Entanglement entropy and $T\bar{T}$ deformations beyond antipodal points from holography. 2019.

[74] S.S. Gubser, I.R. Klebanov, and A.M. Polyakov. Gauge theory correlators from non-critical string theory. *Physics Letters B*, 428(1-2):105–114, May 1998.

[75] Monica Guica. An integrable Lorentz-breaking deformation of two-dimensional CFTs. *SciPost Phys.*, 5(5):048, 2018.

[76] Monica Guica and Ruben Monten. $T\bar{T}$ and the mirage of a bulk cutoff. 2019.

[77] Jutho Haegeman, Tobias J. Osborne, Henri Verschelde, and Frank Verstraete. Entanglement Renormalization for Quantum Fields in Real Space. *Phys. Rev. Lett.*, 110(10):100402, 2013.

[78] Jonathan J. Halliwell and James B. Hartle. Integration Contours for the No Boundary Wave Function of the Universe. *Phys. Rev.*, D41:1815, 1990.

[79] Daniel Harlow. The Ryu–Takayanagi Formula from Quantum Error Correction. *Commun. Math. Phys.*, 354(3):865–912, 2017.

[80] Thomas Hartman, Jorrit Kruthoff, Edgar Shaghoulian, and Amirhossein Tajdini. Holography at finite cutoff with a T^2 deformation. *arXiv:1807.11401[hep-th]*, 2018.

[81] Stephen Hawking, Juan Martin Maldacena, and Andrew Strominger. de Sitter entropy, quantum entanglement and AdS / CFT. *JHEP*, 05:001, 2001.

[82] Song He, Tokiro Numasawa, Tadashi Takayanagi, and Kento Watanabe. Notes on Entanglement Entropy in String Theory. *JHEP*, 05:106, 2015.

[83] Matthew Headrick. Entanglement Renyi entropies in holographic theories. *Phys. Rev.*, D82:126010, 2010.

[84] Idse Heemskerk and Joseph Polchinski. Holographic and Wilsonian Renormalization Groups. *JHEP*, 06:031, 2011.

[85] M. Henningson and K. Skenderis. The Holographic Weyl anomaly. *JHEP*, 07:023, 1998.

[86] Christopher P. Herzog, Kuo-Wei Huang, and Kristan Jensen. Universal Entanglement and Boundary Geometry in Conformal Field Theory. *JHEP*, 01:162, 2016.

[87] Christoph Holzhey, Finn Larsen, and Frank Wilczek. Geometric and renormalized entropy in conformal field theory. *Nucl. Phys.*, B424:443–467, 1994.

[88] Gerard 't Hooft. The holographic principle. *Basics and Highlights in Fundamental Physics*, Apr 2001.

[89] Qi Hu and Guifre Vidal. Spacetime Symmetries and Conformal Data in the Continuous Multiscale Entanglement Renormalization Ansatz. *Phys. Rev. Lett.*, 119(1):010603, 2017.

[90] Ling-Yan Hung, Robert C. Myers, Michael Smolkin, and Alexandre Yale. Holographic Calculations of Renyi Entropy. *JHEP*, 12:047, 2011.

[91] Luca V. Iliesiu, Silviu S. Pufu, Herman Verlinde, and Yifan Wang. An exact quantization of Jackiw-Teitelboim gravity. 2019.

[92] C. Imbimbo, A. Schwimmer, S. Theisen, and S. Yankielowicz. Diffeomorphisms and holographic anomalies. *Class. Quant. Grav.*, 17:1129–1138, 2000.

[93] Eyo Eyo Ita, Chopin Soo, and Hoi-Lai Yu. Intrinsic time gravity, heat kernel regularization, and emergence of Einstein's theory. *arXiv:1707.02720[gr-qc]*, 2017.

[94] Steven Jackson, Razieh Pourhasan, and Herman Verlinde. Geometric RG Flow. *arXiv:1312.6914[hep-th]*, 2013.

[95] Ted Jacobson. A Note on Hartle-Hawking vacua. *Phys. Rev.*, D50:R6031–R6032, 1994.

[96] Yunfeng Jiang. Expectation value of $\mathrm{T}\overline{\mathrm{T}}$ operator in curved spacetimes. *arXiv:1903.07561[hep-th]*, 2019.

[97] Justin Khoury and Herman L. Verlinde. On open - closed string duality. *Adv. Theor. Math. Phys.*, 3:1893–1908, 1999.

[98] Elias Kiritsis, Wenliang Li, and Francesco Nitti. Holographic RG flow and the Quantum Effective Action. *Fortsch. Phys.*, 62:389–454, 2014.

[99] Zohar Komargodski. The $t\bar{T}$ deformation, 2018. Second Simons Bootstrap Collaboration School, Caltech.

[100] Per Kraus, Junyu Liu, and Donald Marolf. Cutoff AdS_3 versus the $T\overline{T}$ deformation. *JHEP*, 07:027, 2018.

[101] K. Kuchar. Geometrodynamics Regained: A Lagrangian Approach. *Journal of Mathematical Physics*, 15:708–715, 1974.

[102] Sung-Sik Lee. Background independent holographic description : From matrix field theory to quantum gravity. *JHEP*, 10:160, 2012.

[103] Sung-Sik Lee. Quantum Renormalization Group and Holography. *JHEP*, 01:076, 2014.

[104] Robert G. Leigh, Onkar Parrikar, and Alexander B. Weiss. Exact renormalization group and higher-spin holography. *Phys. Rev.*, D91(2):026002, 2015.

[105] Aitor Lewkowycz and Juan Maldacena. Generalized gravitational entropy. *JHEP*, 08:090, 2013.

[106] Jennifer Lin. Ryu-Takayanagi Area as an Entanglement Edge Term. 2017.

[107] Hong Liu and Arkady A. Tseytlin. D = 4 superYang-Mills, D = 5 gauged supergravity, and D = 4 conformal supergravity. *Nucl. Phys.*, B533:88–108, 1998.

[108] J. M. Lizana and M. Perez-Victoria. Wilsonian renormalisation of CFT correlation functions: Field theory. *JHEP*, 06:139, 2017.

[109] D. Lovelock. The Einstein tensor and its generalizations. *J. Math. Phys.*, 12:498–501, 1971.

[110] Martin Lüscher. Properties and uses of the Wilson flow in lattice QCD. *JHEP*, 08:071, 2010. [Erratum: JHEP03,092(2014)].

[111] Juan Maldacena. *International Journal of Theoretical Physics*, 38(4):1113–1133, 1999.

[112] Edward A. Mazenc, Vasudev Shyam, and Ronak M. Soni. A $T\bar{T}$ Deformation for Curved Spacetimes from 3d Gravity. *arXiv:1912.09179[hep-th]*, 2019.

[113] Pawel O. Mazur and Emil Mottola. The Gravitational Measure, Solution of the Conformal Factor Problem and Stability of the Ground State of Quantum Gravity. *Nucl. Phys.*, B341:187–212, 1990.

[114] Lauren McGough, Márk Mezei, and Herman Verlinde. Moving the CFT into the bulk with $T\overline{T}$. *JHEP*, 04:010, 2018.

[115] Lauren McGough and Herman Verlinde. Bekenstein-Hawking Entropy as Topological Entanglement Entropy. *JHEP*, 11:208, 2013.

[116] Thomas G. Mertens, Henri Verschelde, and Valentin I. Zakharov. Revisiting non-interacting string partition functions in Rindler space. *Phys. Rev.*, D93(10):104028, 2016.

[117] Thomas G. Mertens, Henri Verschelde, and Valentin I. Zakharov. String Theory in Polar Coordinates and the Vanishing of the One-Loop Rindler Entropy. *JHEP*, 08:113, 2016.

[118] Vincent Moncrief. Reduction of the einstein equations in 2+1 dimensions to a hamiltonian system over teichmüller space. *Journal of Mathematical Physics*, 30(12):2907–2914, 1989.

[119] Emil Mottola. Functional integration over geometries. *J. Math. Phys.*, 36:2470–2511, 1995.

[120] Chitraang Murdia, Yasunori Nomura, Pratik Rath, and Nico Salzetta. Comments on holographic entanglement entropy in TT deformed conformal field theories. *Phys. Rev.*, D100(2):026011, 2019.

[121] Robert C. Myers, Razieh Pourhasan, and Michael Smolkin. On Spacetime Entanglement. *JHEP*, 06:013, 2013.

[122] Robert C. Myers and Aninda Sinha. Holographic c-theorems in arbitrary dimensions. *JHEP*, 01:125, 2011.

[123] Yu Nakayama. $a-c$ test of holography versus quantum renormalization group. *Mod. Phys. Lett.*, A29(29):1450158, 2014.

[124] Yu Nakayama. Scale invariance vs conformal invariance. *Phys. Rept.*, 569:1–93, 2015.

[125] Frank William John Olver. Uniform asymptotic expansions for weber parabolic cylinder functions of large order. *J. Res. Natl. Bur. Stand. B*, 63:131–169, 1959.

[126] H. Osborn. Weyl consistency conditions and a local renormalization group equation for general renormalizable field theories. *Nucl. Phys.*, B363:486–526, 1991.

[127] H. Osborn and G. M. Shore. Correlation functions of the energy momentum tensor on spaces of constant curvature. *Nucl. Phys.*, B571:287–357, 2000.

[128] I. Papadimitriou. Holographic renormalization as a canonical transformation. *J. High Energ. Phys. (2010)*, 2010: 14:708–715, 2010.

[129] Ioannis Papadimitriou. Holographic Renormalization of general dilaton-axion gravity. *JHEP*, 08:119, 2011.

[130] Ioannis Papadimitriou. Lectures on Holographic Renormalization. *Springer Proc. Phys.*, 176:131–181, 2016.

[131] Ioannis Papadimitriou and Kostas Skenderis. AdS / CFT correspondence and geometry. *IRMA Lect. Math. Theor. Phys.*, 8:73–101, 2005.

[132] Andrea Prudenziati. A perturbative expansion for entanglement entropy in string theory. *arXiv:1805.09311[hep-th].*

[133] Lisa Randall and Raman Sundrum. An Alternative to compactification. *Phys. Rev. Lett.*, 83:4690–4693, 1999.

[134] T. Regge and C. Teitelboim. Improved hamiltonian for general relativity. *Physics Letters B*, 53(1):101 – 105, 1974.

[135] Tullio Regge and Claudio Teitelboim. Role of surface integrals in the hamiltonian formulation of general relativity. *Annals of Physics*, 88(1):286 – 318, 1974.

[136] V. A. Rubakov and O. Yu. Shvedov. A Negative mode about Euclidean wormhole. *Phys. Lett.*, B383:258–261, 1996.

[137] Shinsei Ryu and Tadashi Takayanagi. Holographic derivation of entanglement entropy from AdS/CFT. *Phys. Rev. Lett.*, 96:181602, 2006.

[138] K. Kuchar S.A. Hojman and C. Teitelboim. Geometrodynamics Regained. *Annals Phys.*, 96:88–135, 1976.

[139] Vasudev Shyam. Background independent holographic dual to $T\bar{T}$ deformed CFT with large central charge in 2 dimensions. *JHEP*, 10:108, 2017.

[140] Vasudev Shyam. General Covariance from the Quantum Renormalization Group. *Phys. Rev.*, D95(6):066003, 2017.

[141] Vasudev Shyam. Connecting holographic Wess-Zumino consistency condition to the holographic anomaly. *JHEP*, 03:171, 2018.

[142] Vasudev Shyam. Finite Cutoff AdS_5 Holography and the Generalized Gradient Flow. *JHEP*, 12:086, 2018.

[143] F. A. Smirnov and A. B. Zamolodchikov. On space of integrable quantum field theories. *Nucl. Phys.*, B915:363–383, 2017.

[144] Mark Srednicki. Entropy and area. *Phys. Rev. Lett.*, 71:666–669, 1993.

[145] L. Susskind and J. Lindesay. *An introduction to black holes, information and the string theory revolution: The holographic universe.* 2005.

[146] Leonard Susskind. The world as a hologram. *Journal of Mathematical Physics*, 36(11):6377–6396, Nov 1995.

[147] Marika Taylor. TT deformations in general dimensions. *arXiv:1805.10287[hep-th]*, 2018.

[148] Claudio Teitelboim. How commutators of constraints reflect the space-time structure. *Annals Phys.*, 79:542–557, 1973.

[149] Nico M Temme. Numerical and asymptotic aspects of parabolic cylinder functions. *Journal of computational and applied mathematics*, 121(1-2):221–246, 2000.

[150] Arkady A. Tseytlin. On sigma model RG flow, 'central charge' action and Perelman's entropy. *Phys. Rev.*, D75:064024, 2007.

[151] Xueda Wen, Shunji Matsuura, and Shinsei Ryu. Edge theory approach to topological entanglement entropy, mutual information and entanglement negativity in Chern-Simons theories. *Phys. Rev.*, B93(24):245140, 2016.

[152] Wolfgang Wieland. Fock representation of gravitational boundary modes and the discreteness of the area spectrum. *Annales Henri Poincare*, 18(11):3695–3717, 2017.

[153] Edward Witten. (2+1)-Dimensional Gravity as an Exactly Soluble System. *Nucl. Phys.*, B311:46, 1988.

[154] Edward Witten. Anti de sitter space and holography, 1998.

[155] Edward Witten. A Note On Boundary Conditions In Euclidean Gravity. *arXiv:1805.11559[hep-th]*, 2018.

[156] Gabriel Wong. A note on entanglement edge modes in Chern Simons theory. *JHEP*, 08:020, 2018.

[157] Alexander B. Zamolodchikov. Expectation value of composite field T anti-T in two-dimensional quantum field theory. *arXiv:hep-th/0401146*, 2004.

[158] Mengqi Zhu. *$T\overline{T}$ deformations of quantum field theory.* PhD thesis, SISSA, Via Bonomea 265 - 34136 Trieste, Italy, 2018.

APPENDICES

Appendix A

Details about the $D+1$ split

A.1 Gauss and Codazzi relations

In this section, we will provide some useful formulae for the $D+1$ decomposition of the Riemann curvature tensor and its various contractions. These will be needed for the Hamiltonian formulation of General Relativity. For more detailed derivations of these formulae, see for instance [72]. First, we will need the orthogonal projector:

$$\gamma^A_B = \delta^A_B - n^A n_B, \tag{A.1}$$

which will allow us to project tensors in $d = D+1$ dimensions to its tangential components along the D dimensional slices:

$$T^{A_1\cdots A_n}_{||\, B_1\cdots B_n} = \gamma^{A_1}_{E_1}\gamma^{B_1}_{F_1}\cdots\gamma^{A_n}_{E_n}\gamma^{B_n}_{F_n}T^{E_1\cdots E_n}_{F_1\cdots F_n} \tag{A.2}$$

Applying this to vectors, we see a very straightforward relationship:

$$v^A_{||} = \gamma^A_B v^B = v^A - n^A(v^B n_B). \tag{A.3}$$

This straightforwardly generalizes to covariant derivatives of tensors as well:

$$\nabla_{||\, K}T^{A_1\cdots A_n}_{||\, B_1\cdots B_n} = \gamma^{A_1}_{E_1}\gamma^{B_1}_{F_1}\cdots\gamma^{A_n}_{E_n}\gamma^{B_n}_{F_n}\gamma^J_K\nabla_J T^{E_1\cdots E_n}_{F_1\cdots F_n}. \tag{A.4}$$

Then, we also note that the extrinsic curvature tensor can be defined in relation to the normal:

$$\nabla_B n_A = K_{AB} + a_A n_B, \tag{A.5}$$

where $a^A \equiv n^C \nabla_C n^A$, and so

$$(\nabla_B n_A)_{||} = K_{AB}. \tag{A.6}$$

This allows us to write the covariant derivatives of vector fields for instance as:

$$\nabla_A v_B = \nabla_{||\,A} v_{||\,B} + K_{AC} v^C n_B \tag{A.7}$$

With these relations, we can obtain the following projection formula for the Riemann tensor, noting that it can be obtained from the commutator between covariant derivatives and applying the projection relations:

$$\gamma^A_E \gamma^F_B \gamma^G_C \gamma^H_D R^E_{FGH} = R^A_{||\,BCD} - K^A_C K_{BD} + K^A_D K_{BC}. \tag{A.8}$$

This is known as the Gauss relation. Form this relation, we can obtain a contracted Gauss relation:

$$\gamma^E_A \gamma^F_B R_{EF} - \gamma_{AE} n^B \gamma^P_B n^Q R^A_{FPQ} = R_{||AB} - K K_{AB} + K_{AC} K^C_B, \tag{A.9}$$

which can further be contracted till we obtain

$$R - 2R_{AB} n^A n^B = R_{||} - K^2 + K^{AB} K_{AB}. \tag{A.10}$$

Further, we can look at other contractions between the Riemann tensor and the normal vectors:

$$\gamma^C_A n^B \gamma^I_E \gamma^J_F R^A_{BIJ} = \nabla_F K^C_E - \nabla_E K^C_F. \tag{A.11}$$

This is known as the Codazzi relation.

A.2 $D + 1$ split of the Action and the Hamiltonian formalism

Now we consider the decomposition of the Einstein–Hilbert action

$$S = \int \mathrm{d}^d x \sqrt{g} R(g). \tag{A.12}$$

Then, using the decompositions described above, and the relation

$$\sqrt{g} = N \sqrt{g_{||}} \tag{A.13}$$

where, due to the lack of indices, the notation of the parallel symbol is used to denote the determinant of the metric induced on the hypersurface. Then, noting that in fact, the contracted Gauss relation written solely in terms of qunatities tangential to the hypersurface reads

$$R = R_{||} - K^2 + K^{\mu\nu}K_{\mu\nu} + \mathcal{L}_n K + \frac{1}{N}\nabla_\mu \nabla^\mu N, \tag{A.14}$$

and that the total derivatives under the integral can be dropped (here we assume that there are no boundaries of the spacetime region under consideration), the action can now be written in $D+1$ split form:

$$S = \int \mathrm{d}r \int \mathrm{d}^D x N\sqrt{g}\left(R + K^2 - K_{ij}K^{ij}\right). \tag{A.15}$$

Here, the subscript $||$ has been dropped since all the quantities are understood to be tangential to the hypersurfaces. From here, the Legendre transform can be performed.

The relationship between the extrinsic curvature and the 'velocity' with respect to the metric reads

$$K_{\mu\nu} = \mathcal{L}_n g_{\mu\nu} = \dot{g}_{\mu\nu} - \mathcal{L}_\xi g_{\mu\nu}, \tag{A.16}$$

where ξ^μ denotes the shift vector. Then, we define the momentum conjugate to the metric induced on the hypersurface

$$\pi_{\mu\nu} = \frac{\partial L}{\partial \dot{g}_{\mu\nu}} = \frac{1}{N}(K_{\mu\nu} - g_{\mu\nu}K). \tag{A.17}$$

Then, the Hamiltonian can be defined through the Legendre transform

$$H_{tot} = \pi^{\mu\nu}\dot{g}_{\mu\nu} - L, \tag{A.18}$$

This gives us the following expressions:

$$H_{tot} = -\int \mathrm{d}^D x\sqrt{g}NH + \xi^\mu H_\mu, \tag{A.19}$$

where:

$$H(g,\pi) = -\frac{G_{\mu\nu\rho\sigma}}{\sqrt{g}}\pi^{\mu\nu}\pi^{\rho\sigma} - \sqrt{g}R \tag{A.20}$$

$$H_\mu(g,\pi) = -2\nabla_\rho \pi^\rho_\mu. \tag{A.21}$$

A.2.1 2+1 Decomposition in First-Order Formalism

This section of the appendix summarises an application of the ideas in the rest of the appendix to the first-order formalism in 2+1 spacetime dimensions.

We can write the 3d metric in terms of the vielbeins via the standard relation

$$g_{AB}(x) = \delta_{ij} E^i_A(x) E^j_B(x) \tag{A.22}$$

The spin connection is defined as

$$dE^i + \omega^i{}_j \wedge E^j = 0 \tag{A.23}$$

A peculiarity of 3d is that we may define a one indexed spin connection using the Levi-Civita symbol $\omega^i = \varepsilon^{ijk}\omega_{jk}$. In terms of these variables, the Einstein Hilbert action for a 3d spacetime with negative cosmological constant reads (after a rescaling of the vielbeins):

$$S_{3d} = \frac{l}{16\pi G_N}\int_{\mathcal{M}_3} E_i \wedge R(\omega)^i - \frac{1}{6}\varepsilon_{ijk}E^i \wedge E^j \wedge E^k, \quad R^i \equiv d\omega^i + \varepsilon^{ijk}\omega_j \wedge \omega_k, \tag{A.24}$$

where G_N is Newton's constant and l is the AdS_3 radius of curvature.

Consider now a foliation of the 3d geometry by 2d submanifolds, $\mathcal{M}_3 = \Sigma \times \mathbb{R}$. Using a locally adapted coordinate system with normal direction labeled by a coordinate r and coordinates x^μ on the 2d slice, we decompose the vielbeins and spin connections as:

$$\begin{aligned}
E^0 &= E^0_r dr + n_\mu dx^\mu \\
\omega^0 &= \omega^0_r dr + \omega_\mu dx^\mu \\
E^a &= E^a_r dr + f^a_\mu dx^\mu \\
\omega^a &= \omega^a_r dr + \pi^a_\mu dx^\mu .
\end{aligned} \tag{A.25}$$

In terms of these, the action becomes

$$\begin{aligned}
S_{3d} = \frac{l}{16\pi G_N}\int dr \int_\Sigma \, n \wedge \dot{\omega} + f^a \wedge \dot{\pi}_a &+ E_{r,0}\left\{ d\omega + \frac{1}{2}\varepsilon_{ab}\left(\pi^a \wedge \pi^b - f^a \wedge f^b\right)\right\} \\
&+ E^a_r \left\{ d\pi_a - \varepsilon_{ab}\left(\omega \wedge \pi^b - n \wedge f^b\right)\right\} \\
&+ \omega^0_r \left\{ dn - \varepsilon_{ab}\pi^a \wedge f^b \right\} \\
&+ \omega^a_r \left\{ df^a + \varepsilon_{ab}\left(\omega \wedge f^b - \pi^b \wedge n\right)\right\}.
\end{aligned} \tag{A.26}$$

where the dot denotes the partial derivative with respect to the radial coordinate. We may view this as a Hamiltonian system with canonically conjugate variables $\{n_\mu, \varepsilon^{\mu\nu}\omega_\nu\}$

and $\{f^a_\mu, \varepsilon^{\mu\nu}\pi_{a,\nu}\}$. agree with this def of momentum here? because of the wedge The radial components of the vielbeins and of 3d spin connection serve as Lagrange multipliers enforcing constraints. From this form of the action, we see the Hamiltonian consists solely of these constraints C_α, which we label as

$$\begin{aligned}
H &= d\omega + \frac{1}{2}\varepsilon_{ab}\left(\pi^a \wedge \pi^b - f^a \wedge f^b\right) \\
P_a &= d\pi_a - \varepsilon_{ab}(\omega \wedge \pi^b - n \wedge f^b) \\
G &= dn - \varepsilon_{ab}\pi^a \wedge f^b \\
G_a &= df_a + \varepsilon_{ab}(\omega \wedge f^b + n \wedge \pi^b).
\end{aligned} \tag{A.27}$$

Note these constraints C_α are local and hold pointwise on Σ. The H constraint encodes invariance under re-foliations. Its quantization leads to the Wheeler-de-Witt equation. The two P_a constraints correspond to diffeomorphisms tangential to the 2d surface while G generates local Lorentz rotations.

As can be seen from (A.25), the induced metric on Σ is

$$ds^2_{||} = \left(\delta_{ab} f^a_\mu f^b_\nu + n_\mu n_\nu\right) dx^\mu dx^\nu \tag{A.28}$$

. To make contact with the second-order formalism, we can use the G_a constraints to set the redundant variable n_μ to zero; this amounts to orienting the local tangent spaces to agree with the foliation. Following [61], we call this 'radial' gauge. In this gauge the G_a constraints just become the torsionlessness constraint setting ω to be a function of the vielbeins.

Appendix B

Heat Kernel Calculation

B.1 Generalities

The heat kernel $K(x, y; \epsilon)$, satisfies the property

$$\lim_{\epsilon \to 0} K(x, y; \epsilon) = \delta(x, y). \tag{B.1}$$

This property should be thought of as an initial condition for the heat equation

$$\partial_\epsilon K(x, y; \epsilon) = (\nabla^2_{(x)} + \xi R_{(x)}) K(x, y; \epsilon). \tag{B.2}$$

This object admits an expansion in small ϵ:

$$K(x, y; \epsilon) = \frac{\exp\left(\left(-\frac{g_{\mu\nu}(x)}{4\epsilon} - \frac{R_{\mu\nu}(x)}{24}\right)(x-y)^\mu (x-y)^\nu\right)}{(4\pi\epsilon)^{\frac{3}{2}}} \Bigg\{ 1 + \tag{B.3}$$

$$\epsilon a_1(g; \xi) - \epsilon^2 a_2(g; \xi) + O(\epsilon^3) \Bigg\},$$

where:

$$a_1(g; \xi) = \left(\xi - \frac{1}{6}\right) R(x), \tag{B.4}$$

$$a_2(g, \xi) = \frac{1}{6}\left(\xi - \frac{1}{5}\right) \nabla^2 R(x) + \frac{1}{2}\left(\xi - \frac{1}{6}\right)^2 R^2(x) + \frac{1}{60} R^{\mu\nu} R_{\mu\nu}(x) - \frac{1}{180} R^2(x). \tag{B.5}$$

These are known as the Seeley–de Witt coefficients.

B.2 Generating R in the flow equation

The specific ϵ scaling of the coefficients in the expression (3.100) are chosen such that in the limit $\epsilon \to 0$, the following terms vanish[1]

$$\lim_{\epsilon \to 0} C[g] = 0 = \lim_{\epsilon \to 0} \frac{\delta C[g]}{\delta g_{\mu\nu}}. \tag{B.6}$$

The second functional derivative however will remain finite, provided we smear it against the heat kernel. This means that if we distribute the limit, we have

$$\lim_{\epsilon\to 0} \int \mathrm{d}^D y K(x,y,\epsilon) G_{\mu\nu\rho\sigma}(x) \tfrac{1}{\sqrt{g}(x)} \tfrac{\delta}{\delta g_{\mu\nu}(x)} \left(\tfrac{1}{\sqrt{g}(y)} \tfrac{\delta(e^{C[g]} Z[g])}{\delta g_{\rho\sigma}(y)} \right) \tag{B.7}$$

$$= \lim_{\epsilon\to 0} \int \mathrm{d}^D y K(x,y,\epsilon) G_{\mu\nu\rho\sigma}(x) \tfrac{1}{\sqrt{g}(x)} \tfrac{\delta}{\delta g_{\mu\nu}(x)} \left(\tfrac{1}{\sqrt{g}(y)} \tfrac{\delta Z[g]}{\delta g_{\rho\sigma}(y)} \right) \tag{B.8}$$

$$+ \left(\lim_{\epsilon\to 0} \int \mathrm{d}^D y K(x,y,\epsilon) G_{\mu\nu\rho\sigma}(x) \tfrac{1}{\sqrt{g}(x)} \tfrac{\delta}{\delta g_{\mu\nu}(x)} \left(\tfrac{1}{\sqrt{g}(y)} \tfrac{\delta C[g]}{\delta g_{\rho\sigma}(y)} \right) \right) Z[g]. \tag{B.9}$$

We then take a closer look at the term on the second line of the RHS in the expression above

$$\left(\lim_{\epsilon\to 0} \int \mathrm{d}^D y K(x,y,\epsilon) G_{\mu\nu\rho\sigma}(x) \tfrac{1}{\sqrt{g}(x)} \tfrac{\delta}{\delta g_{\mu\nu}(x)} \left(\tfrac{1}{\sqrt{g}(y)} \tfrac{\delta C[g]}{\delta g_{\rho\sigma}(y)} \right) \right) Z[g], \tag{B.10}$$

$$= -\alpha_0 \lim_{\epsilon\to 0} \left(\left(\tfrac{D(D^2-3)}{(D-1)} \right) \tfrac{\epsilon^{\frac{D}{2}-1}}{4} K(x,x;\epsilon) + \tfrac{2\epsilon^{\frac{D}{2}}}{D} \left(\nabla^2_{(x)} + \xi R_{(x)} \right) K(x,x;\epsilon) \right) Z[g] \tag{B.11}$$

Here

$$\xi = -\left(\frac{3D^3 - 4D^2 - 9D + 14}{2D(D-1)} \right). \tag{B.12}$$

Then, the heat equation implies that we can write

$$\lim_{\epsilon\to 0} \alpha_0 \left(\left(\tfrac{D(D^2-3)}{(D-1)} \right) \tfrac{\epsilon^{\frac{D}{2}-1}}{4} K(x,x;\epsilon) + \tfrac{2\epsilon^{\frac{D}{2}}}{d} \left(\nabla^2_{(x)} + \xi R_{(x)} \right) K(x,x;\epsilon) \right) \tag{B.13}$$

$$= \alpha_0 \lim_{\epsilon\to 0} \left(\left(\tfrac{D(D^2-3)}{(D-1)} \right) \tfrac{\epsilon^{\frac{D}{2}-1}}{4} K(x,x;\epsilon) + \tfrac{2\epsilon^{\frac{D}{2}}}{D} \partial_\epsilon K(x,x;\epsilon) \right) = \alpha_0 R(x). \tag{B.14}$$

[1] Note that the order of limits here is to first take $\epsilon \to 0$ with N fixed and then taking $N \to \infty$ at the end.